Lattice Liouville Field Theory in $d = 2$

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Work with:
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Plan of the talk

Liouville Field Theory in $d = 2$

Introduction and motivations

- it is a scalar model in $d = 2$, described by the Euclidean action:

$$S = \frac{1}{\beta^2} \int d^2x \left[-\frac{1}{2} \phi(x) \Box \phi(x) + m^2 e^{\phi(x)} \right]$$

- the field ϕ and the parameter β are dimensionless; m is a mass
- at the classical level, one has invariance of the action under $x^\mu \to e^s x^\mu, \phi \to \phi - 2s$
- while at the quantum mechanical level, $x^\mu \to e^s x^\mu, \phi \to \phi - 2s \cdot (1 + \frac{\beta^2}{8\pi})$

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- Liouville Field Theory (LFT) arises in string theory in various situations;
- Recent work suggests that LFT might have possible applications to high energy QCD as an effective theory for gluon phenomenology (see Iancu and McLerran, 2007);
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The Lattice Model

To put the theory on the lattice, we start by defining the Euclidean action

$$S(\beta, m^2, L, \phi) = \frac{1}{\beta^2} \sum_{x \in R_L} \left[-\frac{1}{2}\phi(x)\Box\phi(x) + m^2 e^{\phi(x)} \right]$$

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- L is the number of lattice sites in each direction
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a few more comments:

- finite volume and finite lattice spacing break scale invariance explicitely;
- scale invariance is a subgroup of conformal invariance;
- the minimum for the classical action is at $\phi \to -\infty$ which is not compatible with Monte Carlo methods: two approaches are possible

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First approach: **constrained field**

the field is decomposed

$$\phi(x) = \phi_0 + \chi(x)$$
$$\sum_x \chi(x) = 0$$

which implies

$$S(\beta, m^2, L, \phi) \quad = \quad S(\beta, m^2 e^{\phi_0}, L, \chi)$$

a change in the average field ϕ_0 is equivalent to a change of m. The simulation then corresponds to a model in which $[d\chi]$ only is integrated in the functional integrals, leaving ϕ_0 free.

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Second approach: **point source**

a current interaction is added to the action with a point source

$$S_{\mathrm{curr}} = \frac{\alpha}{\beta^2} \sum_{x \in R_L} \phi(x) \delta(x - x_0)$$

which has a stable minimum configuration (found by numerical analysis, $\alpha \in [0, 1]$)
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- the constrained field method is not used to evaluate observables but only to investigate the continuum limit;
- both approaches are related to the true Liouville theory by analytical relations
- in particular, the second approach can be used to evaluate $n-$point functions of operators of the form

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Exact results

Some exact results have been obtained:

Partition functions (exact):

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\begin{aligned}
Z(\beta, m^2, L) &= \int D\chi d\phi_0 \, e^{-S(\beta, m^2 e^{\phi_0}, L, \chi)} = \\
&= \int d\phi_0 \, Z_c(\beta, m^2 e^{\phi_0}, L) = \\
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Results for $e^{\alpha\phi}$ (exact):

$$\left\langle e^{\alpha\phi} \right\rangle_c = \frac{Z_c(\beta, m^2, L, \alpha\beta^2\delta_x)}{Z_c(\beta, m^2, L)}$$

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$$\left\langle e^{\alpha\phi} \right\rangle (\beta, m^2, L) = $$

$$= \frac{1}{Z(\beta, L)} \int \frac{d\tau^2}{\tau^2} \left(\frac{\tau^2}{m^2}\right)^{\alpha} Z_c \left\langle e^{\alpha\chi(x)} \right\rangle_c$$

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Approximated results

The following results hold at first order in an m^2 expansion.
We need to define the following function:

$$g(L) = -\frac{1}{L^2} tr\Box^{-1}$$

$$g(L) = \frac{1}{2\pi} \ln\left(\frac{L}{L_c}\right) + O\left(\frac{1}{L^2}\right)$$

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we also calculated the correlation length at first order in m^2:

$$\frac{\xi^2(\beta, m^2, L)}{L^2} =$$

$$= \frac{\beta^2}{m^2 L^2} \left(\frac{L}{L_0}\right)^{-\frac{\beta^2}{4\pi}} \left[1 + m^2 L^2 \left(\frac{L}{L_0}\right)^{\frac{\beta^2}{4\pi}} f(L)\right]$$

where $f(L)$ is finite in the $L \to \infty$ limit.

The following RG transformation

$$\begin{aligned} x &\rightarrow e^s x \\ L &\rightarrow e^s L \\ m^2 &\rightarrow e^{-2\left(1+\frac{\beta^2}{8\pi}\right)s} m^2 \equiv e^{-2\gamma s} m^2 \end{aligned} \tag{1}$$

leaves unchanged

$$Z_c, \frac{\xi}{L}, \beta$$

A fixed point is found at

$$L \rightarrow \infty, m \rightarrow 0$$

around which we will try to construct a continuum limit for the unconstrained field theory. The perturbative origin of these results should be kept in mind.

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Along an *RG* trajectory specified by some m_0, L_0, the following result holds

$$\langle e^{\chi} \rangle_c (\beta, m^2, L) = A \left(\frac{L}{L_0} \right)^{2(\gamma-1)}$$

$$\gamma = 1 + \frac{\beta^2}{8\pi}$$

Continuum Limit

- we need to verify the scaling properties of 1 and 2 point functions
- along the RG trajectories in the $L \to \infty$ limit
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2−point functions

$$\left\langle e^{\alpha\phi(x)} e^{\alpha\phi(y)} \right\rangle$$

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- field configurations where generated with a Hybrid Monte Carlo algorithm
- simulated volumes from $L = 120$ to $L = 240$
- one configuration in 100 saved to reduce correlations
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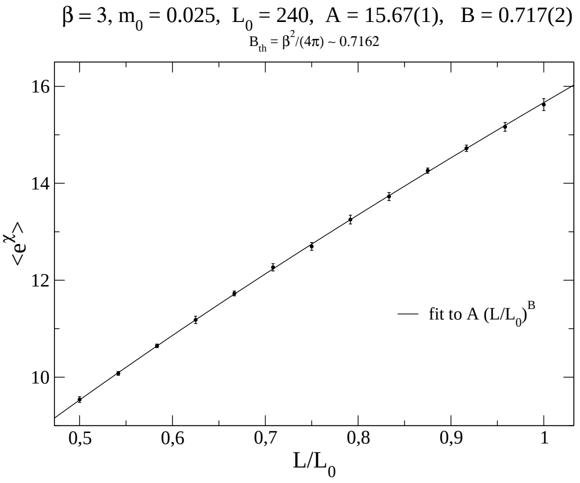
Simulation results for $\langle e^{\chi} \rangle_c$



As a first estimate of the error on the numerical value of γ, we tried to deform the RG trajectory by

$$m^2 \to e^{-2gs} m^2$$

where g differs from the analytic result by a given percentage. We tried deformations of

$$10, 5, 2.5, 1, 0.5$$

percent: the corresponding fits gave results for the exponent which where compatible with the expected result only for deformations of 0.5%.

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Conclusions and outlook

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- no proof of continuum limit but good clues that it might work out right (scaling properties of observables)
- many more simulations need to be performed
- many questions need answers: continuum limit, scale invariance...

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