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A Fully Numerical Approach to One-Loop Amplitudes

M. Moretti, F.P., A.D. Polosa, arXiv:0802.4171 [hep-ph]

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1. Motivations
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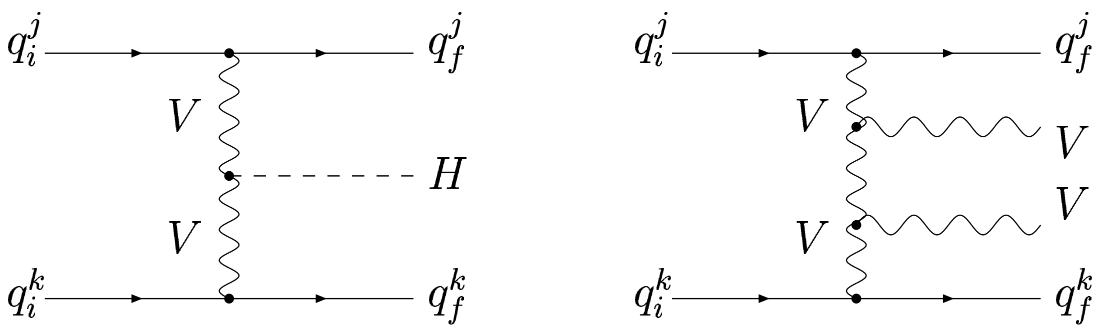
SM&FT, Bari, 03/09/2008

Motivations

- At the LHC (NLC) processes with many HARD particles in the final state will play a crucial role.

1. top quark production and decay
2. associated weak bosons productions (higgs searches, structure of the EW interactions)
3. weak bosons fusion and weak boson scattering



4. Large center of mass energy $\Rightarrow$ Drell-Yan and prompt photon associated to a large number of hard jets.
5. heavy resonances (SUSY, compositness, ...)

- Several tools for the evaluation of complex Matrix Element (ME), at the leading order approximation, are avaliable. ME element with more than TEN external particles are avaliable and all these tools are interfaced with parton showers and hadronization tools for a realistic description of the event.

  ALPGEN M.L. Mangano, M. Moretti, F. Piccinini, R. Pittau and A.D. Polosa, JHEP **0307** (2003) 001.

  HELAC A. Kanaki and C.G. Papadopoulos, Comput. Phys. Commun. **132** (2000) 306, [arXiv:hep-ph/0002082]; C.G. Papadopoulos and M. Worek, Eur. Phys. J. **C50** (2007) 843, [arXiv:hep-ph/0512150].

  MADEVENT T. Stelzer and W.F. Long, Comput. Phys. Commun. **81** (1994) 357, [arXiv:hep-ph/9401258]; F. Maltoni and T. Stelzer, JHEP **02** (2003) 027, [arXiv:hep-ph/0208156].

  SHERPA T. Gleisberg, S. Höche, F. Krauss, A. Schälicke, S. Schumann and J. Winter, , JHEP 0402 (2004) 056, [arXiv:hep-ph/0311263].

  HELICITY H. Murayama, I. Watanabe and K. Hagiwara, "Helas: Helicity amplitude subroutines for Feynman diagram evaluations", KEK-91-11 (1992).

NLO computation, recent progress

Standard approach: dimensional regularization + Passarino Veltman techniques. Difficulties: factorially growing number of terms to be evaluated (and occasionally problems in the numerical evaluation of Passarino Veltman functions). Several highly demanding calculations already performed, e.g.:

$pp \to VVV$ A. Lazopoulos, K. Melnikov and F. Petriello, Phys. Rev. **D76** (2007) 014001, [arXiv:hep-ph/0703273];

V. Hankele and D. Zeppenfeld, arXiv:0712.3544 [hep-ph];

T. Plehn and M. Rauch, Phys. Rev. **D72** (2005) 053008, [arXiv:hep-ph/0507321];

T. Binoth, S. Karg, N. Kauer and R. Ruckl, Phys. Rev. **D74** (2006) 113008, [arXiv:hep-ph/0608057].

$pp \to H\ j\ j\ (m_t \to \infty)$ J.M. Campbell, R.K. Ellis and G. Zanderighi, JHEP **0610** (2006) 028, [arXiv:hep-ph/0608194];

M.M. Weber, Nucl. Phys. Proc. Suppl. **160** (2006) 200.

$pp \to \bar{t}\ t\ j$ S. Dittmaier, P. Uwer and S. Weinzierl, Phys. Rev. Lett. **98** (2007) 262002, [arXiv:hep-ph/0703120].

$pp \to W(Z)\ W\ j\ j$ B. Jager, C. Oleari and D. Zeppenfeld, JHEP **0607** (2006) 015, [arXiv:hep-ph/060317]

G. Bozzi, B. Jager, C. Oleari and D. Zeppenfeld, Phys. Rev. **D75** (2007) 073004;

$e^+e^- \to 4$ fermions A. Denner, S. Dittmaier, M. Roth and L.H. Wieders, Phys. Lett. **B612** (2005) 223, [arXiv:hep-ph/0502063];

A. Denner, S. Dittmaier, M. Roth and L.H. Wieders, Nucl. Phys. **B724** (2005) 247, [arXiv:hep-ph/0505042].

$gg \to gggg$ R.K. Ellis, W.T. Giele and G. Zanderighi, JHEP **05** (2006) 027, [arXiv:hep-ph/0602185].

Advance in complexity: String inspired QCD Feynman rules, unitarity, twistor inspired recursion relations

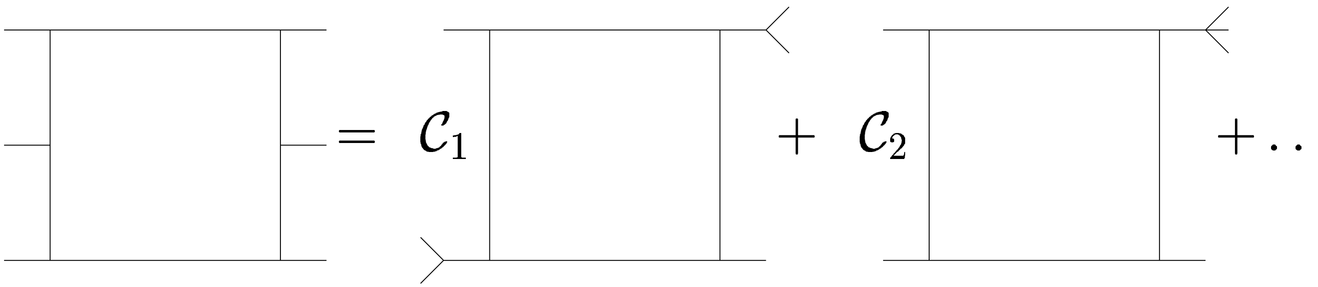
$\gamma\gamma \to \gamma\gamma\gamma\gamma$ G. Mahlon, Phys. Rev. **D49** (1994) 2197, [arXiv:hep-ph/9311213]; arXiv:hep-ph/9412350; Z. Nagy and D.E. Soper, Phys. Rev. **D74** (2006) 093006, arXiv:hep-ph/0610028; G. Ossola, C.G. Papadopoulos and R. Pittau, JHEP **0707** (2007) 085, arXiv:0704.1271 [hep-ph]; T. Binoth, T. Gehrmann, G. Heinrich and P. Mastrolia, Phys. Lett. **B649** (2007) 422, arXiv:hep-ph/0703311.

Review of cut-construction of diagram and recursion relation for rational part of the diagrams Z. Bern, L.J. Dixon, D.A. Kosower, Ann. Phys. **322** (2007) 1587, arXiv:0704.2798 [hep-ph].

Efficient algorithm for the numerical evaluation of the coeffients G. Ossola, C.G. Papadopoulos and R. Pittau, Nucl. Phys. **B763** (2007) 147 [arXiv:hep-ph/0609007]; JHEP **0707** (2007) 085, [arXiv:0704.1271[hep-ph]]; arXiv:0711.3596[hep-ph].

Algorithms for the computation of the rational part of the diagram T. Binoth, J.P. Guillet, G. Heinrich, E. Pilon and C. Schubert, JHEP **0510** (2005) 015, [arXiv:hep-ph/0504267]; T. Binoth, J.P. Guillet and G. Heinrich, JHEP **0702** (2007) 013, [arXiv:hep-ph/0609054]; Z.G. Xiao, C. Yang and C.J. Zhu, Nucl. Phys. **B758** (2006) 1, [arXiv:hep-ph/0607015]; X. Su, Z.G. Xiao, C. Yang and C.J. Zhu, Nucl. Phys. **B758** (2006) 35, [arXiv:hep-ph/0607016]; Z. Bern, L.J. Dixon and D.A. Kosower, Phys. Rev. **D73** (2006) 065013, [arXiv:hep-ph/0507005]; S.D. Badger, E.W.N. Glover and K. Risager, JHEP **0707** (2007) 066, [arXiv:0704.3914[hep-ph]]; C.F. Berger, Z. Bern, L.J. Dixon, D. Forde and D.A. Kosower, Phys. Rev. **D75** (2007) 016006, [arXiv:hep-ph/0607014]; C. Anastasiou, R. Britto, B. Feng, Z. Kunszt and P. Mastrolia, Phys. Lett. **B645** (2007) 213 [arXiv:hep-ph/0609191]; JHEP **0703** (2007) 111; R. Britto and B. Feng, arXiv:hep-ph/07114284.

Basically



very much the same as

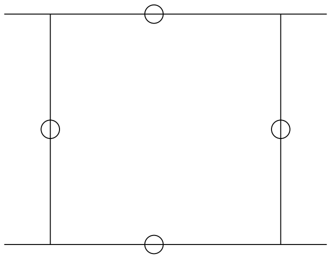
$$\frac{Ax + B}{(x - x_1)(x - x_2)(x - x_3)} = \sum_{j=1}^{3} \mathcal{C}_j/(x - x_j)$$

since our x_j are four-vectors

$$\frac{1}{x - x_j} \to \text{product of four propagators}$$

Actually, bubbles and triangles required as well, this implies non trivial problems

To pick up the residues $\mathcal{C}_j$ evaluate the functions on the poles: four propagators on shell $\Rightarrow$ complex momenta. In this way one can profit of the maior advantage of recursion techniques for tree-level amplitudes: power law growth in computational complexity as opposed to the factorial growth of the number of Feynman graphs.



The $\mathcal{C}_j$ are thus evaluated as convolution of four tree-level amplitudes (with complex external four momenta)

very recently a numeric approach to the above ideas suitable for authomatic evaluation of the required coefficients

Algorithm W.T. Giele, Z. Kunszt and K. Melnikov, arXiv:0801.2237 [hep-ph].

TWENTY gluons NLO amplitude W.T. Giele and G. Zanderighi, arXiv:0805.2152 [hep-ph]

Let's finally mention Numerical evaluation of individual Feynman graphs

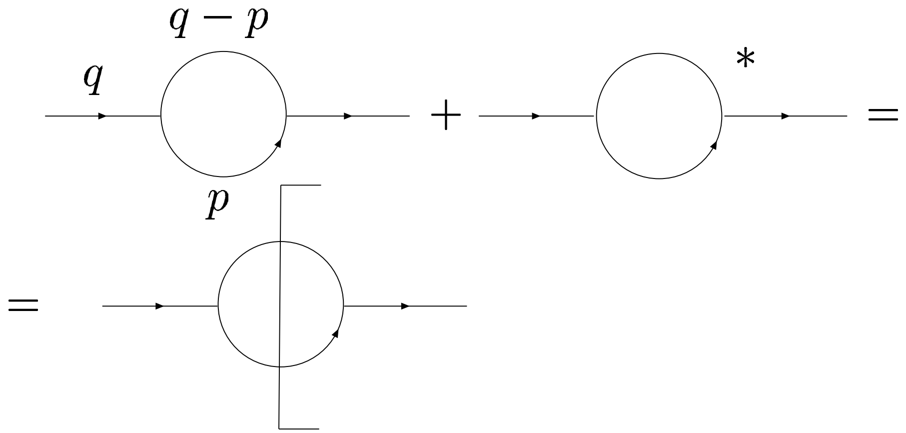
Avoiding large denominators A. Ferroglia, M. Passera, G. Passarino and S. Uccirati, Nucl. Phys. **B650** (2003) 162, arXiv:hep-ph/0209219; G. Passarino, Nucl. Phys. bf B619 (2001) 257, arXiv:hep-ph/0108252.

Application to two-loop EW corrections S. Actis and G. Passarino, Nucl. Phys. **B777** (2007) 100, hep-ph/0612124; Nucl. Phys. **B777** (2007) 35, hep-ph/0612123; S. Actis, A. Ferroglia, M. Passera and G. Passarino, Nucl. Phys. **B777** (2007) 1.

Complex integration to remove the integral over dl_0 Z. Nagy and D.E. Soper, JHEP **0309** (2003) 055, [arXiv:hep-ph/0308127]; Acta Phys. Polon. **B35** (2004) 2557; M. Kramer and D.E. Soper, Phys. Rev. **D66** (2002) 054017, [arXiv:hep-ph/0204113]; D.E. Soper, Phys. Rev. Lett. **81** (1998) 2638, [arXiv:hep-ph/9804454]; Phys. Rev. **D62** (2000) 014009, [arXiv:hep-ph/9910292]; Phys. Rev. **D64** (2001) 034018, [arXiv:hep-ph/0103262].

Our proposal

Take advantage of our ability to tackle arbitrairly complex tree-level matrix element and reformulate the problem as a suitable convolution of products of tree-level matrix element.



From the imaginary part of the diagram, via dispersion relations, to the computation of the whole diagram.

$$\mathrm{Real}[G_2(p)] \sim \int \mathrm{d}s \frac{\mathrm{Im}[G_2(s)]}{(s-p^2)}$$

We finally aim to a tool for the authomatic computation of NLO amplitude. We thus need a relation which is independent from the specific process.

To this end we shall make use of a fairly old dispersion-like relation established by T. Veltman to study unitarity and causality for a individual Feynman graphs.

M.J. Veltman, “Dispersive calculation of diagrams with arbitrarily external legs”, in, “Colloquium on Advanced Computing Methods in Theoretical Physics”, vol. 2, pag. IV-115, Marseille, 21-25 June 1971; G.’t Hooft and M.J. Veltman, “Diagrammar,” CERN Report 73-79, 1973, published in NATO Adv. Study Inst. Ser. B Phys. 4 p. 177-322, 1974

The largest time equation

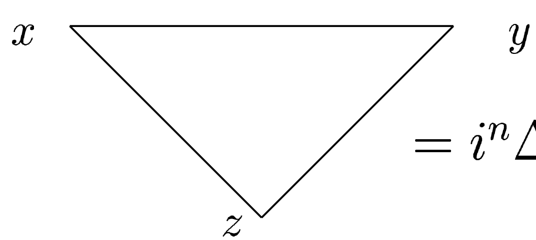
let's consider a scalar theory for simplicity

The propagator for a scalar field is

$$\Delta_{xy} = \int \frac{\mathrm{d}l}{(2\pi)^4} \mathrm{e}^{-il(x-y)} \frac{i}{l^2 - m^2} = \theta(x_0 - y_0)\Delta^+_{xy} + \theta(y_0 - x_0)\Delta^+_{yx}$$

$$\Delta^+_{xy} = \int \frac{\mathrm{d}l}{(2\pi)^3} \mathrm{e}^{-il(x-y)} \theta(l_0)\delta(l^2 - m^2)$$

For a generic n-point Feynman diagram G_n



$$= i^n \Delta_{xy}\Delta_{yz}\Delta_{zx}$$

We now introduce a new notation: the "underlined configuration variable" $\underline{x}_j$ and a new set of functions

$$\tilde{G}_3(x, \underline{y}, z) = (-1)^{n_u} i^3 \Delta_{x\underline{y}} \Delta_{\underline{y}z} \Delta_{zx}$$

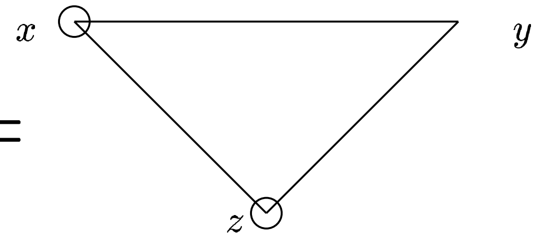
n_u(=1 here) is the number of underlyined variables and

$$\begin{aligned}
\Delta_{\underline{x}_l x_m} &= \Delta^+_{x_m x_l} = \int \frac{\mathrm{d}l}{(2\pi)^3} \mathrm{e}^{-il(x_l - x_m)} \theta(l_0) \delta(l^2 - m^2) \\
\Delta_{x_l \underline{x}_m} &= \Delta^+_{x_l x_m} \\
\Delta_{\underline{x}_l \underline{x}_m} &= \Delta^*_{x_l x_m}
\end{aligned}$$

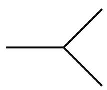
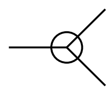
Notice that because of the $\theta(l_0)$ term *energy is always required to flow towards the underlined vertex*

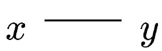
Diagrammatic notation for $\tilde{G}_n$: a small circle for a vertex with an "attached" underlined variable:

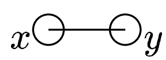
$$\tilde{G}(\underline{x}, y, \underline{z}) =$$



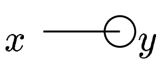
Therefore the set of Feynman rules for an underlined diagrams are:

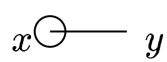
 $\equiv i;$  $\equiv -i$



$\equiv \Delta_{xy};$ 

$\equiv \Delta^{*}_{xy};$

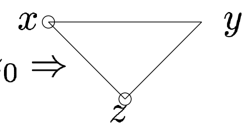


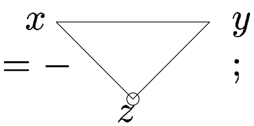
$\equiv \Delta^{+}_{xy};$ 

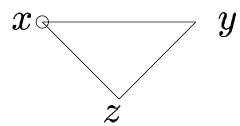
$\equiv \Delta^{+}_{yx}$

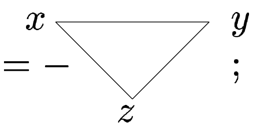
From the previous rules ($x_0 > y_0, z_0 \Rightarrow \Delta_{xy} = \theta(x_0 - y_0)\Delta^+_{xy}$)

$$x_0 > y_0, z_0 \Rightarrow \tilde{G}(x,y,z) = -\tilde{G}(\underline{x},y,z); \quad \tilde{G}(x,\underline{y},z) = -\tilde{G}(\underline{x},\underline{y},z); \quad \ldots$$

$x_0 > y_0, z_0 \Rightarrow$ 

$= -$ 

; 

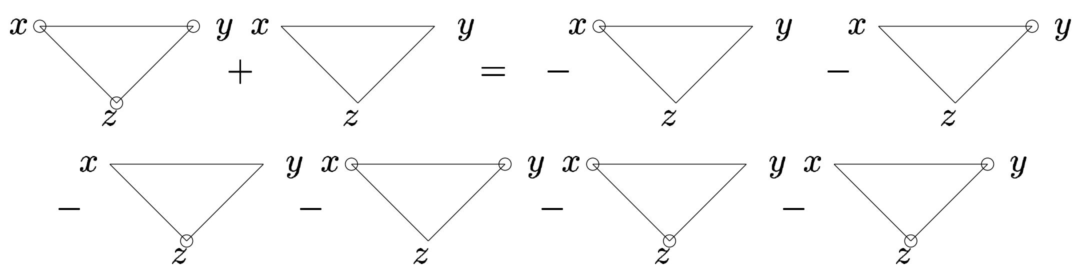
$= -$ 

; ...

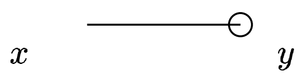
If x_0 is the largest time the sum of a diagram with $\underline{x}$ and of the same diagram with x vanishes $\Rightarrow$ *largest time equation*

$$\tilde{G}(x,y,z) + \tilde{G}(\underline{x},y,z) + \tilde{G}(x,\underline{y},z) + \tilde{G}(x,y,\underline{z}) + \tilde{G}(\underline{x},\underline{y},z)$$
$$+\tilde{G}(\underline{x},y,\underline{z}) + \tilde{G}(x,\underline{y},\underline{z}) + \tilde{G}(\underline{x},\underline{y},\underline{z}) = 0$$

From which the standard Cutkosky rules are recovered



- $\tilde{G}(x,y,z) = \tilde{G}^{*}(\underline{x},\underline{y},\underline{z}) = G(x,y,z)$
- A line connecting a circled and an uncircled vertex is a *cut line*



$$\equiv \Delta^{+}_{xy} = \int \frac{\mathrm{d}l}{(2\pi)^3} \mathrm{e}^{-il(x-y)} \theta(l_0) \delta(l^2 - m^2)$$

- many of the above diagrams vanish because of energy conservation: if a massive particle is attached to the x vertex and massless particles are attached to the y and z vertices (and massless particles flow inside the loop) only the last diagram on the right-hand side is non vanishing.

Dispersion relations

the scalar self-energy case

$$\begin{aligned} G(x,y) &= \Delta_{xy}\Delta_{yx} = \theta(x_0-y_0)(\Delta^+_{xy})^2 + \theta(y_0-x_0)(\Delta^+_{yx})^2 \\ &= \theta(x_0-y_0)\int \mathrm{d}^4k\ \mathrm{d}^4q\, \mathrm{e}^{-i(x-y)(k+q)}\theta(k^0)\vartheta(q^0)\delta(k^2-m_1^2)\delta(q^2-m_2^2) + \theta(y_0-x_0)\ldots \end{aligned}$$

using integral representation for the θ function

$$\theta(x) = \frac{1}{2\pi i}\int \frac{\mathrm{e}^{i\tau x}}{\tau - i\epsilon}\, \mathrm{d}\tau$$

upon Fourier transform

$$\mathrm{e}^{-i(x_0-y_0)(k_0+q_0-\tau)} \to \delta(\tau - p_{20} + k_0 + q_0)]$$

$$\begin{aligned} G(p_1,p_2) &= \frac{1}{(2\pi)^3 i}(2\pi)^4\delta(p_1+p_2) \\ & \int \mathrm{d}\tau \int \mathrm{d}^4k\ \mathrm{d}^4q \frac{1}{\tau - i\epsilon}\delta^3(p_2+k+q)\vartheta(k^0)\vartheta(q^0)\delta(k^2-m_1^2)\delta(q^2-m_2^2) \\ \times & \quad [\delta(\tau+p_{20}+k_0+q_0) + \delta(\tau - p_{20}+k_0+q_0)] \\ &= \frac{1}{(2\pi)^3 i}(2\pi)^4\delta(p_1+p_2)\int \mathrm{d}\tau \frac{1}{\tau - i\epsilon}\int \mathrm{d}^3k \frac{1}{2k_0}\frac{1}{2q_0}[\delta(\tau+p_{20}+k_0+q_0)+\delta(\tau-p_{20}+k_0+q_0)] \end{aligned} \quad (1)$$

Summarizing

- Dispersive integral $\mathrm{d}\tau \frac{1}{\tau}$
- two body phase space integral
  $\mathrm{d}^3k \frac{1}{2k_0} \frac{1}{2q_0} [\delta(\tau + p_{20} + k_0 + q_0) + \delta(\tau - p_{20} + k_0 + q_0)]$
- $\mathrm{d}\tau \frac{1}{\tau - i\epsilon} \to \mathcal{P}\frac{1}{\tau}$ (imaginary part) + $\delta(\tau)$ (real part)
- Principal value integral → dispersion relation for the diagram imaginary part (amplitude real part)
- A $\pm\tau$ amount of energy is flowing into the two vertices "connected" by $\theta(x_0 - y_0)$

Generalizing the dispersion relation. Three point function.

Start from a trivial identity:

$$G(x,y,z) = G(x,y,z)[\theta(y_0 - z_0) + \theta(z_0 - y_0)]$$

From the largest time equation

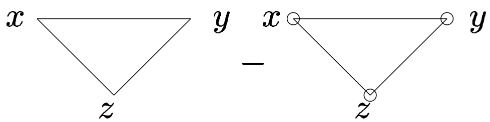
$$\theta(y_0 - z_0)\left[\tilde{G}(x,y,z) + \tilde{G}(x,\underline{y},z) + \tilde{G}(\underline{x},y,z) + \tilde{G}(\underline{x},\underline{y},z)\right] = \theta(y_0 - z_0)[I + II + III + IV] = 0$$

since, if $y_0 > x_0, z_0$ $I + II = 0$ and $III + IV = 0$, if $x_0 > y_0, z_0$ $I + III = 0$ and $II + IV = 0$.

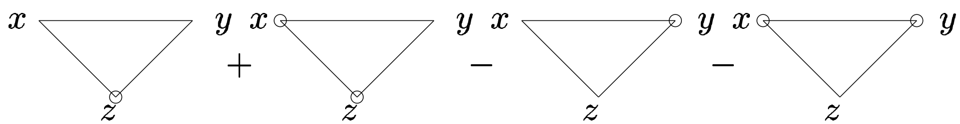
From the above relationship (and the analogous one for the $\sim \theta(z_0 - y_0)$ term

$$\begin{aligned} G(x,y,z) &= -\theta(y_0 - z_0)\left[\tilde{G}(\underline{x},y,z) + \tilde{G}(x,\underline{y},z) + \tilde{G}(\underline{x},\underline{y},z)\right] \\ &\quad -\theta(z_0 - y_0)\left[\tilde{G}(\underline{x},y,z) + \tilde{G}(x,y,\underline{z}) + \tilde{G}(\underline{x},y,\underline{z})\right] \\ &= -\theta(y_0 - z_0)\left[\tilde{G}(x,\underline{y},z) + \tilde{G}(\underline{x},\underline{y},z)\right] - \theta(z_0 - y_0)\left[\tilde{G}(x,y,\underline{z}) + \tilde{G}(\underline{x},y,\underline{z})\right] \\ &\quad -\tilde{G}(\underline{x},y,z) \end{aligned}$$

For twice the imaginary part of the diagram we obtain



$$= [\theta(y_0 - z_0) - \theta(z_0 - y_0)] \Bigg($$



$$\Bigg)$$

- Again each uncircled to circled line denote a “cut” propagator
- Again the combination of θ factors lead to a dispersive $\mathcal{P}\frac{1}{\tau}$ integral times a two body phase space integral
- Again an energy flow $\pm\tau$ occurs in the y and z vertex
- The three-point diagram is cutted into two tree-level like (connected) diagrams in all possible way, provided x and y lie on the opposite side of the cut.
- Notice that any other pair of vertices could have been chosen

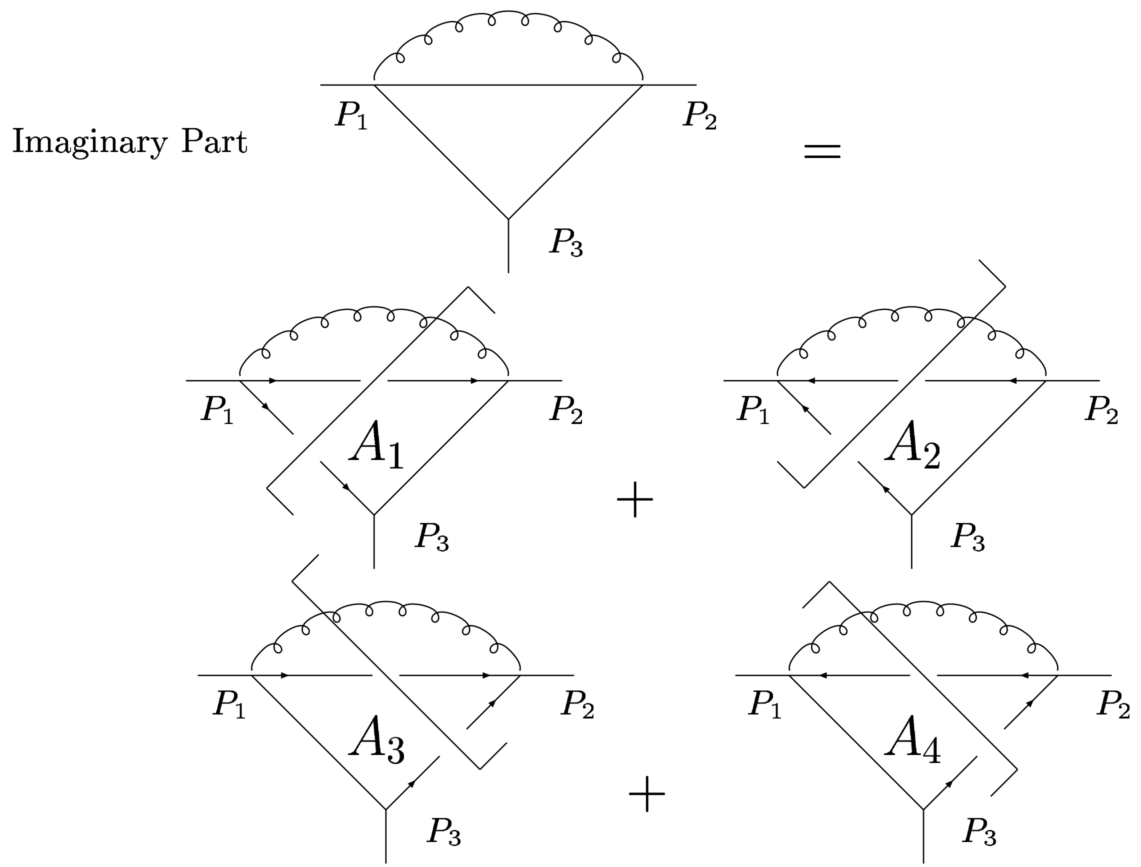


Figure 1: Three point Green function in a scalar ϕ^3 theory. (spring-shaped) line denotes the dispersive integral over dτ as well as internal two body phase space. The arrows on internal cut lines depict the energy flow in the cut diagram.

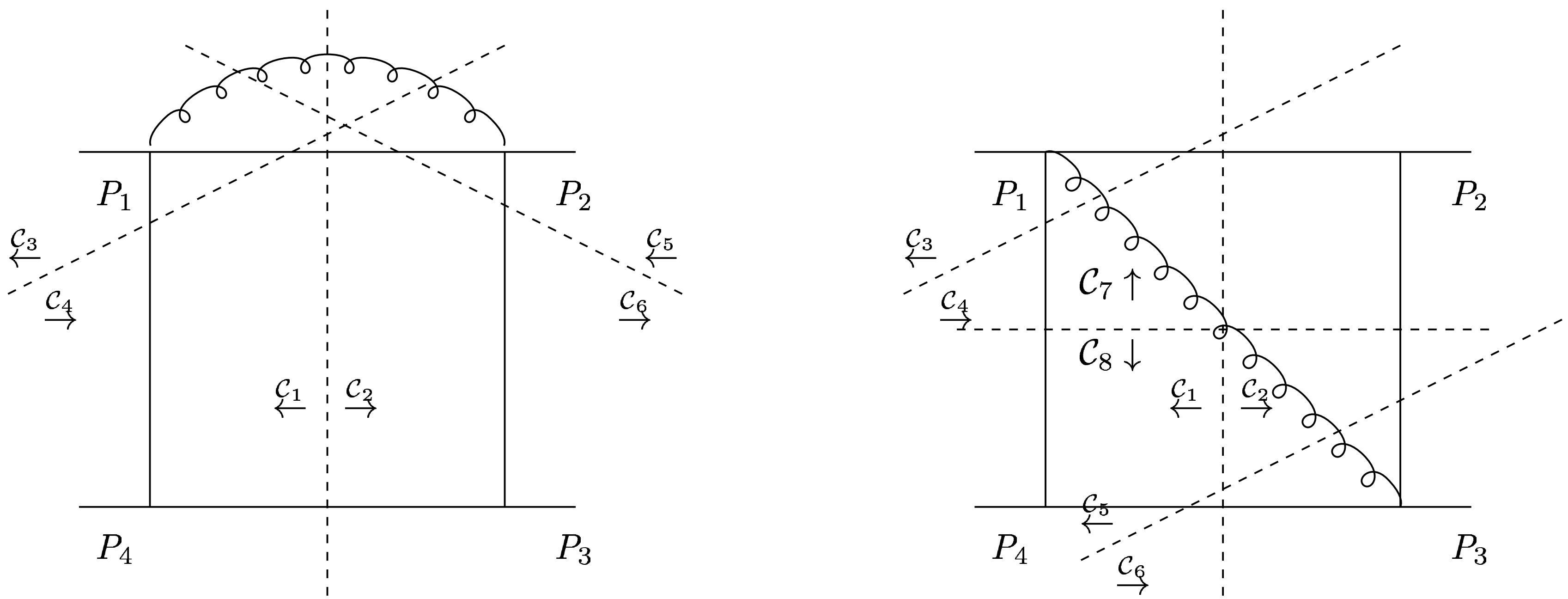


Figure 2: Four point function (Imaginary Part). There are six possible ways to draw the τ-line: four as shown in the left-hand side of the figure (the other three possibilities are similar but with the τ-line among P_2 and P_3 or P_3 and P_4 or P_4 and P_1) and two as shown in the right-hand side of the figure (the other one is similar but with the τ-line among P_2 and P_4). The left hand choice gives the sum of six contributions corresponding to having the propagator connecting P_1 and P_2 cut together with anyone of the remaining propagators. The right hand choice gives eight contributions corresponding to having a cut propagator between P_1 and P_2 or between P_2 and P_3, together with the propagator between P_3 and P_4 or between P_4 and P_1. Recall that each "cut" actually corresponds to two contributions depending on the energy flow.

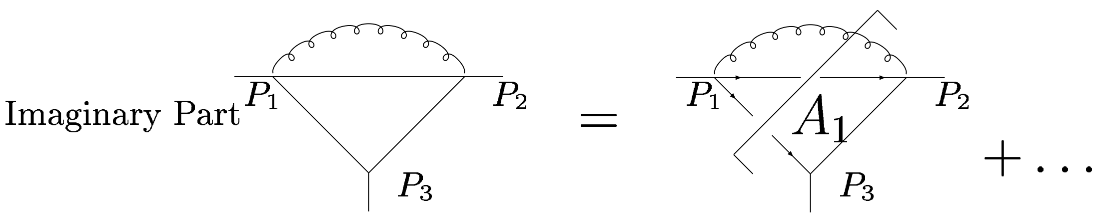
For a generic Green function the method can be applied as well and, summarizing the results sketched above, leads to the following modified Feynman rules:

- Draw a $\tau-$line (the θ function insertion) between two arbitrary vertixes of the Feynman graph.
- Cut the graph in two disconnected parts in such a way that the $\tau-$line crosses (if both the chosen vertices are on the same side of the cut this will contribute only to the real part of the graph) the cut.
- In the vertices where the $\tau-$line is absorbed/emitted it does contribute to the energy balance with a τ.
- The $\tau-$line contributes a factor $1/(i\pi) \times \mathcal{P}1/\tau$ integrated over τ.
- Cut internal lines contribute a factor $\theta(\pm k^0)\delta(k^2 - m^2)$ where $\pm k^0$ is chosen in such a way that their energy flows towards the shaded region.
- A two-body phase space integral is associated with the two cut internal lines: $d\Phi = \lambda^{1/2}(P^2, p_1^2, p_2^2)/P^2 \; d\cos\theta d\phi$ where $p_{1,2}$ are the momenta of the cut internal lines, and P is the sum of the external 4-momenta k_j which enter into the unshaded part of the graph plus the $\tau-$line contribution which amounts to subtract τ to the time-like P component.
- Sum over the contributions of all allowed cuts (see the items above).

Insofar this is Veltman approach to study unitarity and causality of individual Feynman diagrams

We can go one step further summing up all individual diagrams and rewriting the NLO amplitude as convolution of tree-level amplitudes

Let's go back to the three point function



A_1 is the convolution of two tree-level Feynman diagram

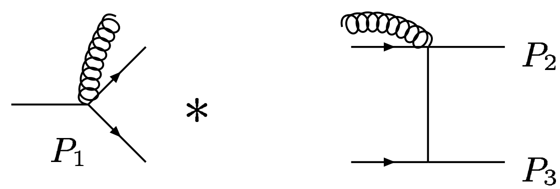


Figure 3: Convolution of Feynman graphs. line denote the ("cut") τ-line, arrows on the "cut" internal lines denote energy flow and $*$ denote the convolution over τ and two body phase space

We can go one step further: convolution of amplitudes rather than Feynman graphs

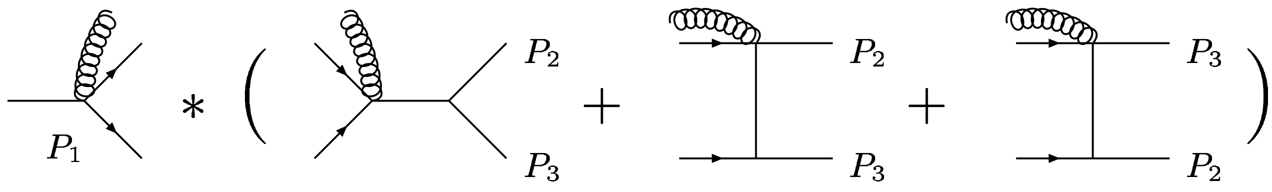
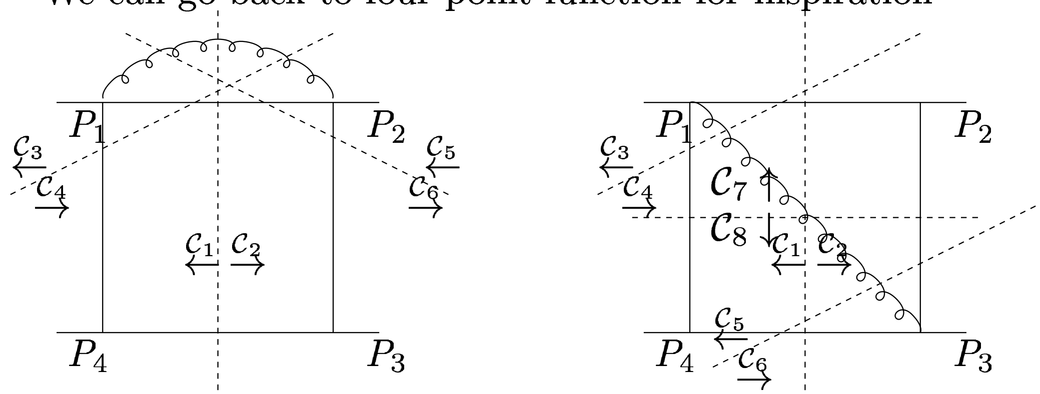


Figure 4: Convolution of amplitudes rather than individual Feynman graphs. It is manifest that, in addition to vertex contributions, we pick up also self energies contributions.

- summing up over the four cuts one get the whole amplitude
- self energies and triangle, however, come out with different combinatorial factors
- for a large number of external particles the choice of the vertices where τ-line is emitted/absorbed becomes ambiguos

We can go back to four point function for inspiration



A way to select the topology to the left is *to require that the τ-line is emitted/absorbed at the opposite end of a single propagator: namely it will always be emitted/absorbed toghether with one "cut"-line, carrying momentum K_1.*

An n-point green function will be computed n-times: still a combinatorial problem

To fix combinatorics we introduce a reference momentum (say P_1 for definitness) and we require that P_1 and K_1 are attached to the same vertex or that K_1 is attached to a tree of external particle including P_1

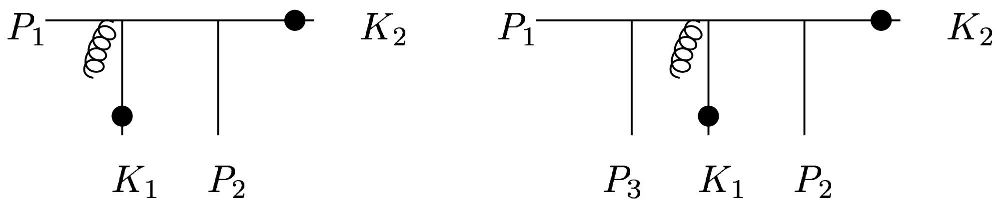


Figure 5: Allowed particles tree into the amplitudes entering into the convolution. Cut propagators (denoted with a small circle) carry momentum K_j and external particles carry momentum P_j. The τ-line is always emitted/absorbed from a vertex connected either directly to P_1 (left-hand side) or to a tree made up of external particles only (K_2 not allowed) and containing P_1. Equivalently, if the tree is seen from the opposite side the τ-line is always emitted/absorbed from a vertex connected to a tree made up of external particles plus K_2 and not containing P_1. The left-hand side diagram, inside the convolution, will contribute to the evaluation of boxes and triangles, whereas the right-hand side to triangles only.

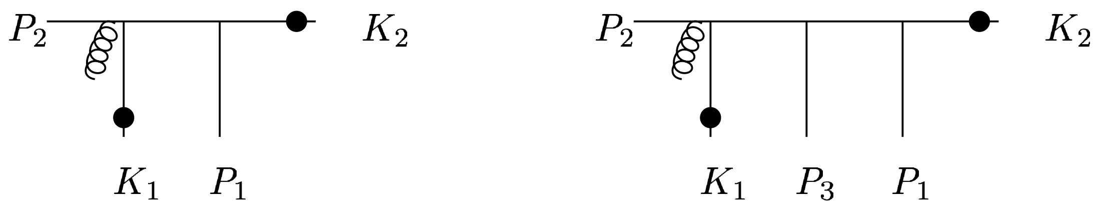
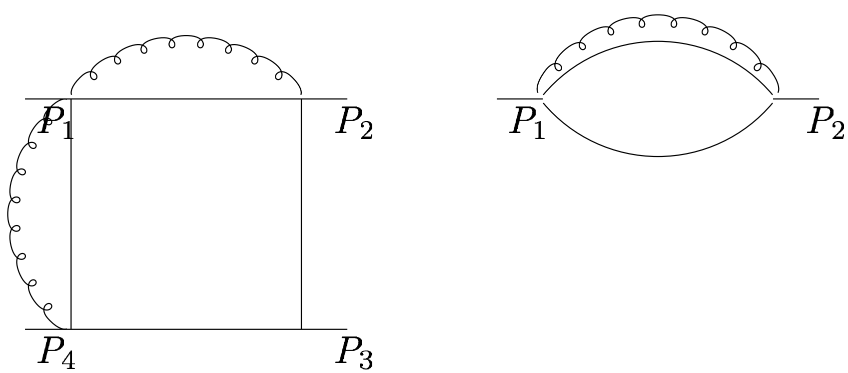


Figure 6: Vetoed particles tree into the amplitudes entering into the convolution. Cut propagators (denoted with a small circle) carry momentum K_j and external particles carry momentum P_j. The K_2 particle is always emitted/absorbed from a vertex connected either directly to P_1 (left-hand side) or to a tree made up of esternal particle only (K_1 not allowed) and containing P_1. Equivalently, if the tree is seen from the opposite side the K_2 particle is always emitted/absorbed from a vertex connected to a tree made up of esternal particles plus K_1 and not containing P_1.



Box ($n(> 2)$-points) counted twice, self-energy once: correct combinatorics

The recipe: consider the amplitude

$$\phi(P_1)\phi(P_2) \to \phi(P_3)\phi(P_4)$$

P_j: momenta of the external particles

- Divide the set of external particles into two non empty subsets $\mathcal{G}_{\alpha_j}$ $j = 1, 2$. There are $14 = \sum_{i=1}^{3} \begin{pmatrix} 4 \\ i \end{pmatrix}$ such partitions.
- Compute the convolution of

$$[\mathcal{G}_{\alpha_1} \to \phi(K_1)\phi(K_2)] * [\phi(K_1)\phi(K_2) \to \mathcal{G}_{\alpha_2}]$$

  summing over all possible partitions. K_j: momenta of the cut lines.
- Require that the τ line is emitted/absorbed at the opposite ends of the cut-line carrying K_1 momentum.
- Select an external momentum (arbitrarily). For definiteness we choose P_1. We will call it reference momentum from here on.

- If the reference momentum belongs to $\mathcal{G}_{\alpha_2}$ the amplitude for $\mathcal{M}_2 = \phi(K_1)\phi(K_2) \to \mathcal{G}_{\alpha_2}$ needs to be modified as follows:
  - If a diagram has a propagator depending on a momentum $Q = \sum_j Q_j$, where $Q_j \neq K_1$ and at least P_1 and K_2 enter the sum over j, this diagram must be vetoed.
  - If a diagram has a propagator containing K_1 and at least an external P_j but not P_1, this diagram must be vetoed.
  - The diagram of $\mathcal{M}_2$ with K_1 and K_2 attached to the same vertex should be vetoed if $\mathcal{G}_{\alpha_1}$ is made up of a single external particle. This prescription allows us to avoid to compute external particle self energies. This contributions will be computed analytically and added in a second step.
- The above prescriptions will apply also to $\mathcal{G}_{\alpha_1} \to \phi(K_1)\phi(K_2)$ if P_1 belongs to $\mathcal{G}_{\alpha_1}$.

Let us examine one of such partitions:

$$\mathcal{G}_{\alpha_1} = \{\phi(P_4)\}; \quad \mathcal{G}_{\alpha_2} = \{\phi(P_1), \phi(P_2), \phi(P_3)\}$$

The calculation of the convolution $\mathcal{M}_{\alpha_1} * \mathcal{M}_{\alpha_2}$ for such partition is sketched in the figure.

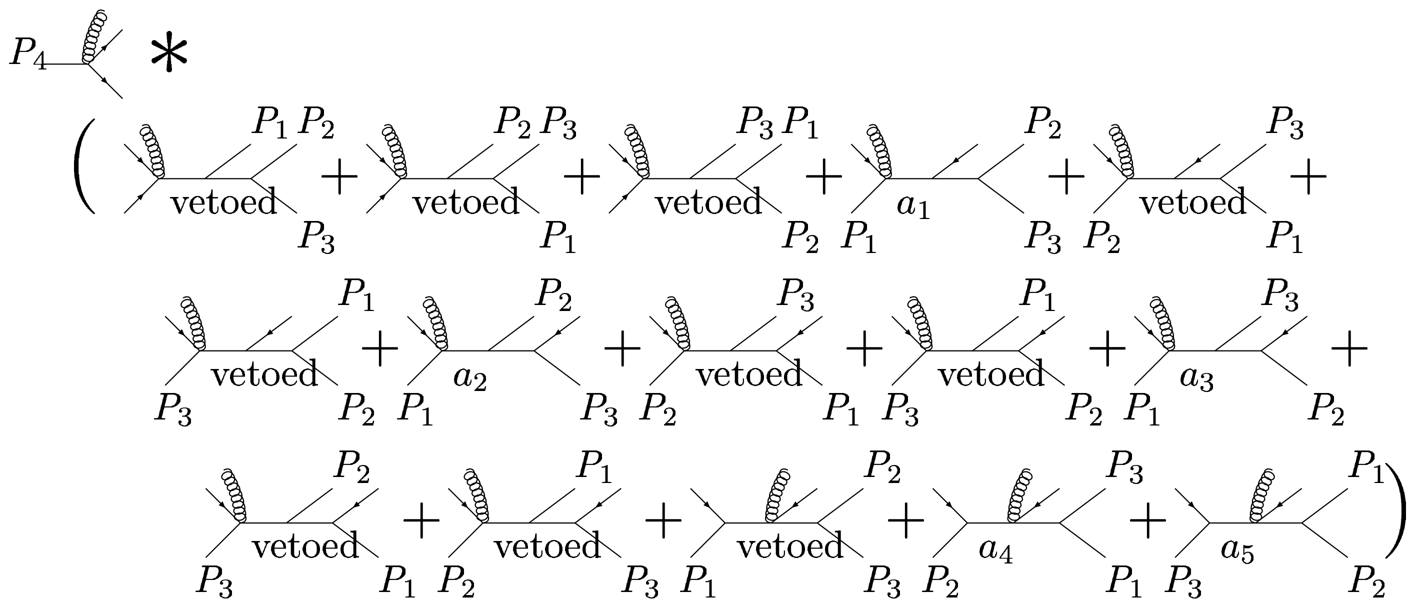


Figure 7: Cut lines carry an arrow. indicates a τ-line. The non-vetoed graphs correspond to different one-loop diagrams: a_1, a_4 and a_5 are one of the cuts of three different triangles, whereas a_2 and a_3 contribute to two different box diagrams.

Numerical checks and conclusions

Principal value integrals:

$$\mathcal{P}\frac{1}{x}f(x) \to \frac{1}{x}f(x) - \frac{1}{x}f(0) + \mathcal{P}\frac{1}{x}f(0)$$

(the location of the poles is easily found)

The scalar three-point function

	p_x (GeV)	p_y (GeV)	p_z (GeV)
p_1	1.73205081	1.41421354	−2.23606801
p_2	−1.73205081	−1.41421354	2.23606801
p_3	0	0	0
	1.982(2) (numerical)	1.98390995 (LoopTools)	

Table 1: Comparison between the values of the scalar form factor C_0 for the ϕ^3 theory as obtained with the numerical method and with the analytical results of LoopTools (T. Hahn and M. Perez-Victoria, Comput. Phys. Commun. **118** (1999) 153.). The momenta components are taken to be $p = (p_x, p_y, p_z)$, with a common internal and external mass of 0.01 GeV.

The scalar four-point function

	p_x (GeV)	p_y (GeV)	p_z (GeV)
p_1	1.73205081	1.41421354	−2.23606801
p_2	−1.73205081	−1.41421354	2.23606801
p_3	−2.23606801	−1.41421354	−1.73205081
p_4	2.23606801	1.41421354	1.73205081
	0.4435(2) (numerical)	0.443615191 (LoopTools)	

Table 2: Comparison between the values of the scalar form factor D_0 for the ϕ^3 theory as obtained with the numerical method and with the analytical results of LoopTools. The momenta components are taken to be $p = (p_x, p_y, p_z)$, with a common internal and external mass of 0.01 GeV.

The scalar five-point function

	p_x (GeV)	p_y (GeV)	p_z (GeV)
p_1	0	0	2.13816766718398
p_2	0	0	-2.13816766718398
p_3	-0.825139760971069	-0.878521561622620	$-0.0866711735725402$8
p_4	-0.501121819019318	-0.772821187973022	-0.162467777729034
p_5	1.32626157999039	1.65134274959564	0.249138951301575
	E_0	-10.48(2) (numerical)	-10.4724461 (LoopTools)

Table 3: Comparison between the values of the scalar form factors for the ϕ^3 theory as obtained with the numerical method and with the analytical results of LoopTools. The momenta components are taken to be $p = (p_x, p_y, p_z)$, with a common internal and external mass of 0.01 GeV.

The cross section for $e^+e^- \to q\bar{q}$

fermion mass (GeV)	analytical (nb)	numerical (nb)
1	-1.0622(2)	-1.0623(3)
2	-0.7879(1)	-0.7882(3)
3	-0.5985(1)	-0.5989(2)
8	+0.01320(0)	+0.01289(8)
9	+0.10271(0)	+0.10258(6)

Table 4: Values of the virtual QCD cross section for the process $e^+e^- \to \gamma^* \to q\bar{q}$, at $\sqrt{s} = 20$ GeV, for different fermion masses. The infrared divergence has been regulated with a photon mass $m_\gamma = 30$ MeV.

We deal with the renormalization problem as follows:

- We compute the 3-point function using a hard cut-off $\tau_{\max}$.
- For a fixed set of momenta compute the 3-point Green function using both the hard cut-off $\tau_{\max}$, obtaining $G_3^{(\tau_{\max})}$, and dimensional regularization (DR), obtaining $G_3^{(DR)}$.
- We then added to the Lagrangian the counterterm $G_3^{(DR)} - G_3^{(\tau_{\max})} = \bar{\psi}(a_0^{(\tau_{\max})}\gamma_0 A_0 - a^{(\tau_{\max})}\gamma \cdot \mathbf{A})\psi$.

Notice: counterterms not Lorentz invariant

The above result has been obtained with a suitably modified version of the `ALPHA` (F. Caravaglios, M. Moretti, Phys. Lett. **B358** (1995) 332.) code for the authomatic evaluation of tree-level ME. As input only external particles and all possible intermediate particles for the convolution.

Still a long way to useful result

- Check numerical stability for more complex case
- Work out the full set of required counterterms
- Find a suitable four-dimensional infra-red cut-off (eventually ugly counterterms are not a problem here since everything is taken care numerically) and extablish the correspondence with DR results to be able to use existing PDF's