

# CHARACTERIZATION OF STATIONARY NON-EQUILIBRIUM STEADY STATES

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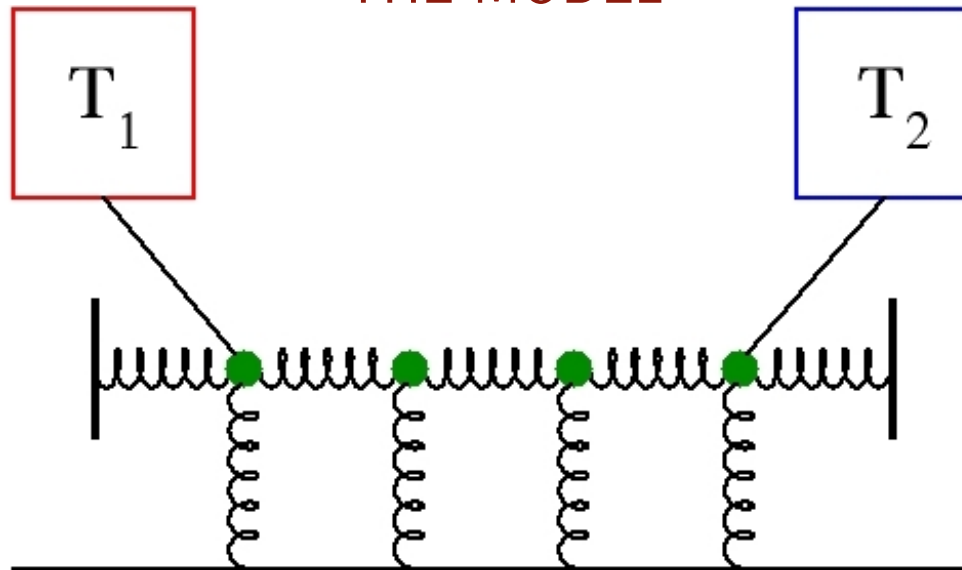
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# OUTLINE

- MODELS AND DEFINITIONS
- ANOMALOUS VS. NORMAL TRANSPORT IN 1D SYSTEMS
- A SOLVABLE STOCHASTIC MODEL
- ANALYSIS OF THE STATIONARY REGIME
- “NORMAL” MODES

## THE MODEL



$$H = \sum_{i=1}^N \left[ \frac{p_i^2}{2m_i} + V(q_{i+1} - q_i) + U(q_i) \right],$$

$m_i$  particle mass

$V(\cdot)$  potential energy of internal interactions

$U(\cdot)$  on-site potential

## HEAT FLUX

$$j_i = \frac{1}{2}(q_{i+1} - q_i)(p_{i+1} + p_i)f(q_{i+1} - q_i) + p_i h_i$$

$$j_i = \frac{a}{2}(p_i + p_{i+1})f_{i+1} \quad (\text{in the limit of small fluctuations})$$

$a =$  lattice spacing

$f_{i+1} = f(q_{i+1} - q_i) =$  force due to the  $i \rightarrow (i + 1)$  particle interaction

$$j_i = \frac{1}{2}p_i^3 \quad (\text{for an ideal gas})$$

## THERMAL CONDUCTIVITY

$$\kappa(L) = \frac{\text{average heat flux}}{\text{thermal gradient}}$$

normal behaviour

$$\kappa(L)_{\lim L \rightarrow \infty} \quad \text{finite}$$

sub/super conductive behaviour ( $\alpha < 0, > 0$ )

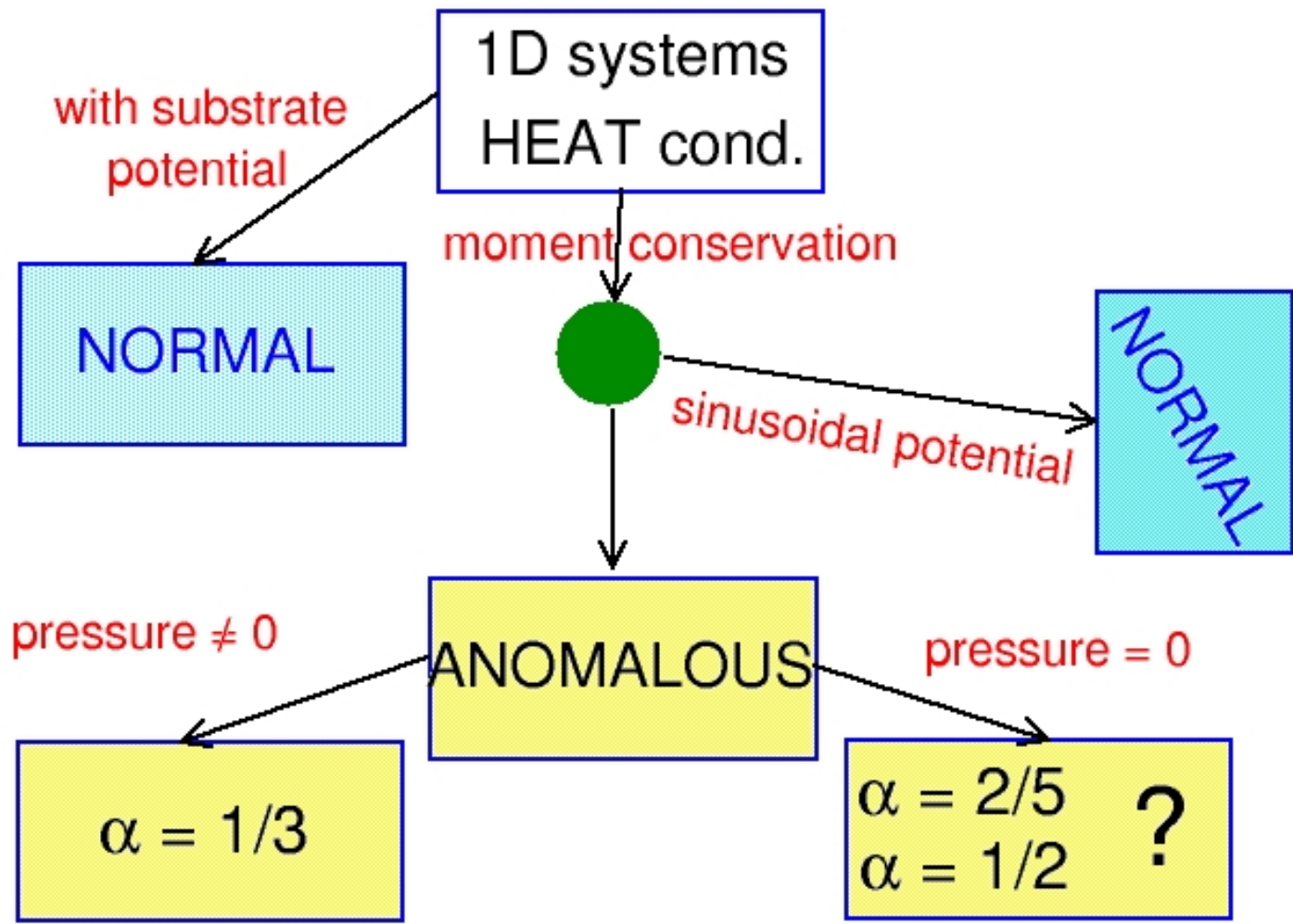
$$\kappa(L)_{\lim L \rightarrow \infty} \approx L^\alpha$$

# THEORETICAL TOOLS

DYNAMICAL Renormalization Group

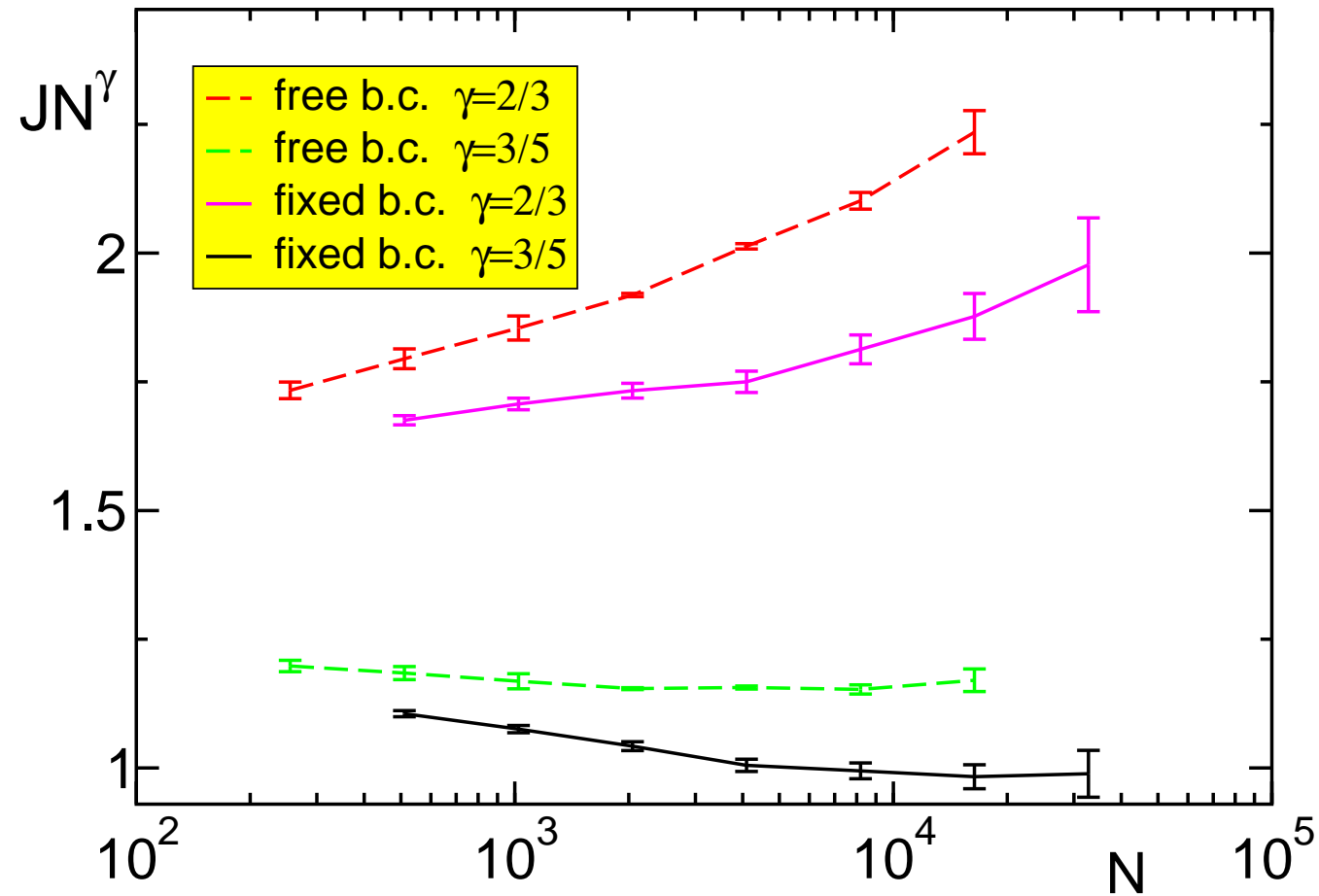
BOLTZMANN-PEIERLS EQUATION

SELF-CONSISTENT MODE COUPLING EQUATIONS



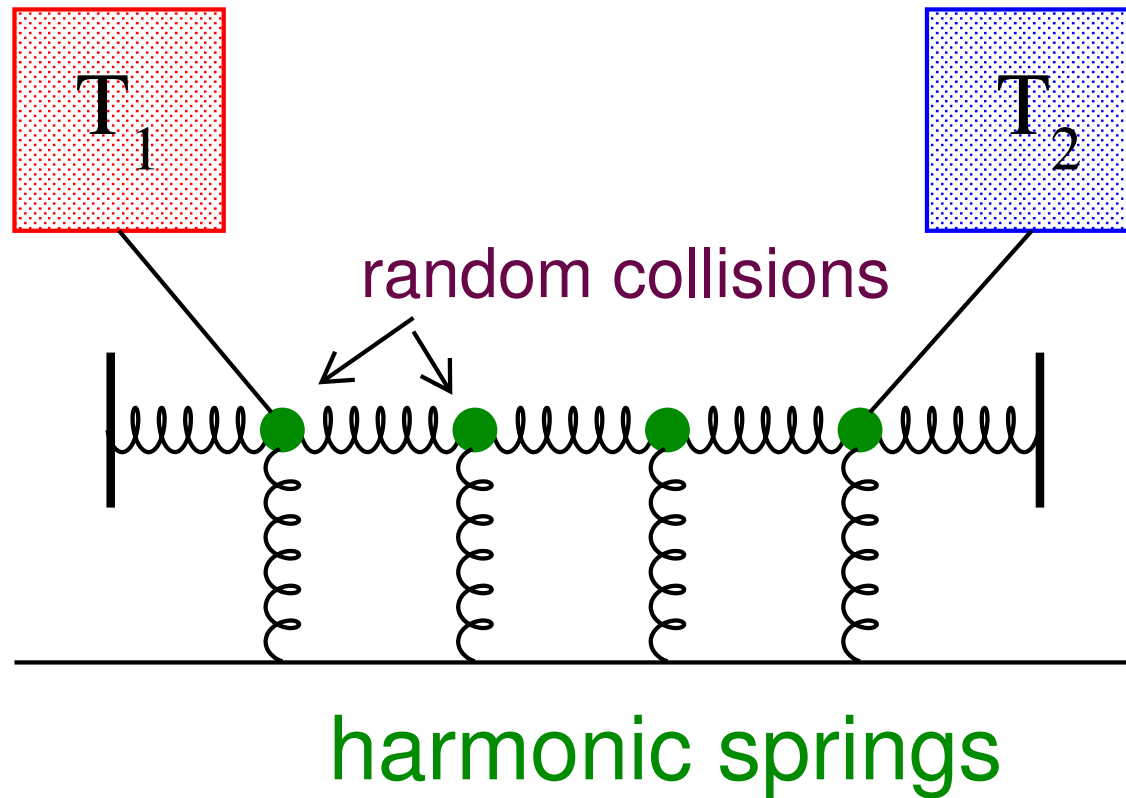
# AN ACCURATE NUMERICAL TEST

FPU- $\beta$  :  $T_+ = 1.1$  ;  $T_- = 0.9$





A STOCHASTIC MODEL  
ADAPTED FROM Basile Bernardin Olla



Kipnis, Marchioro, Presutti 1982

Bernardin, Olla 2005

Giardinà, Kurchan, Redig 2007

## EQUATION FOR THE PROBABILITY DENSITY

$$\frac{\partial P}{\partial t}(x, t) = (\mathbb{L} + \mathbb{L}_{col}) P(x, t) \quad x \equiv (q, p)$$

$$\mathbb{L}P = \sum_{\mu, \nu} \left[ a_{\mu\nu} \frac{\partial}{\partial x_\mu} (x_\nu P) + \frac{d_{\mu\nu}}{2} \frac{\partial^2 P}{\partial x_\mu \partial x_\nu} \right]$$

$$\mathbf{a} = \begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \omega^2 \mathbf{g} + k\mathbf{I} & \lambda \mathbf{r} \end{pmatrix}$$

$$\mathbf{d} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 2\lambda k_B T (\mathbf{r} + \eta \mathbf{s}) \end{pmatrix}$$

$$T = (T_+ + T_-)/2 \quad \eta = (T_+ - T_-)/T$$

$$\mathbb{L}_{col}P = \gamma \sum_i [P(\dots p_{i+1}, p_i \dots) - P(\dots p_i, p_{i+1} \dots)]$$

Covariance matrix

$$c_{\mu\nu} = \langle x_\mu x_\nu \rangle \equiv \int dx P(x, t) x_\mu x_\nu$$

$$\mathbf{c} = \begin{pmatrix} \mathbf{u} & \mathbf{z} \\ \mathbf{z}^\dagger & \mathbf{v} \end{pmatrix}$$

$$\dot{\mathbf{c}} = \mathbf{d} - \mathbf{a}\mathbf{c} - \mathbf{c}\mathbf{a}^\dagger + \dot{\mathbf{c}}_{col} \quad .$$

The contribution due to collisions reads

$$\dot{\mathbf{c}}_{col} = -\gamma \begin{pmatrix} \mathbf{0} & \mathbf{z}\mathbf{g} \\ \mathbf{g}\mathbf{z}^\dagger & \mathbf{w} \end{pmatrix}$$

where the auxiliary  $N \times N$  matrix  $\mathbf{w}$  is defined by

$$w_{ij} \equiv \begin{cases} v_{i+1j} + v_{i-1j} + v_{ij-1} + v_{ij+1} - 4v_{ij} & |i - j| > 1 \\ v_{i\pm 1j} + v_{ij\mp 1} - 2v_{ij} & i - j = \pm 1 \\ v_{i-1j-1} + v_{i+1j+1} - 2v_{ij} & i = j \end{cases}$$

## Stationary solution

$$\mathbf{z}^\dagger = -\mathbf{z}$$

$$\mathbf{v} = \omega^2 \mathbf{u} \mathbf{g} + k \mathbf{u} + \lambda \mathbf{z} \mathbf{r} + \gamma \mathbf{z} \mathbf{g}$$

$$\omega^2 (\mathbf{g} \mathbf{z} + \mathbf{z}^\dagger \mathbf{g}) + \lambda (\mathbf{r} \mathbf{v} + \mathbf{v} \mathbf{r}) + \gamma \mathbf{w} = 2\lambda k_B T (\mathbf{r} + \eta \mathbf{s})$$

Temperature profile:

$$T_i = \langle p_i^2 \rangle = v_{ii}$$

Energy current:

$$J = J_i^d + J_i^c$$

$$J_i^d = \omega^2 \langle q_{i-1} p_i \rangle = \omega^2 z_{i-1,i}$$

$$J_i^c = \frac{\gamma}{2} [\langle p_i^2 \rangle - \langle p_{i-1}^2 \rangle] = \frac{\gamma}{2} [v_{i,i} - v_{i-1,i-1}]$$

## *COORDINATE TRANSFORMATION*

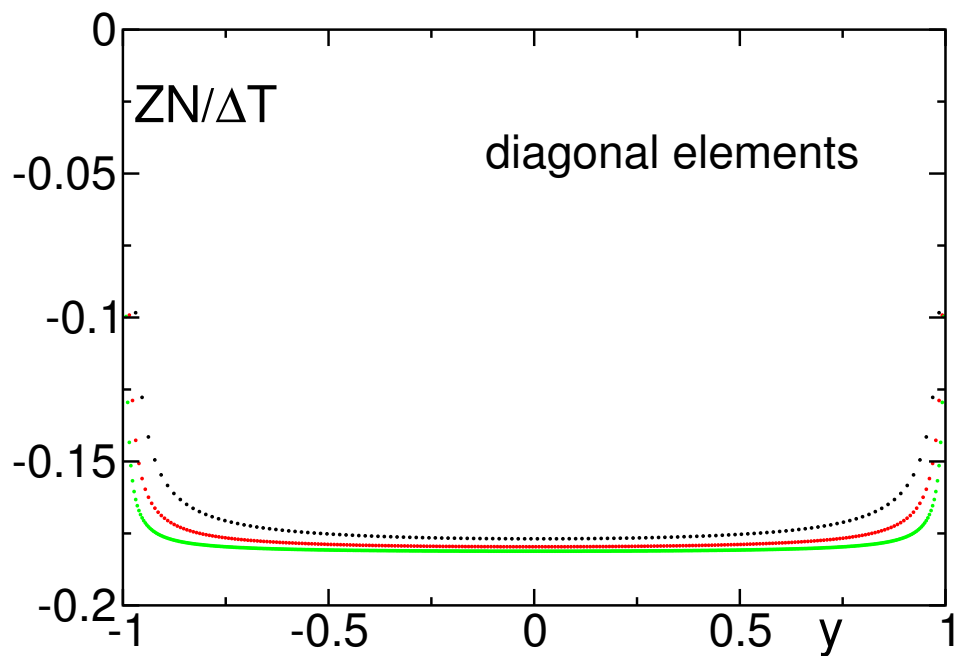
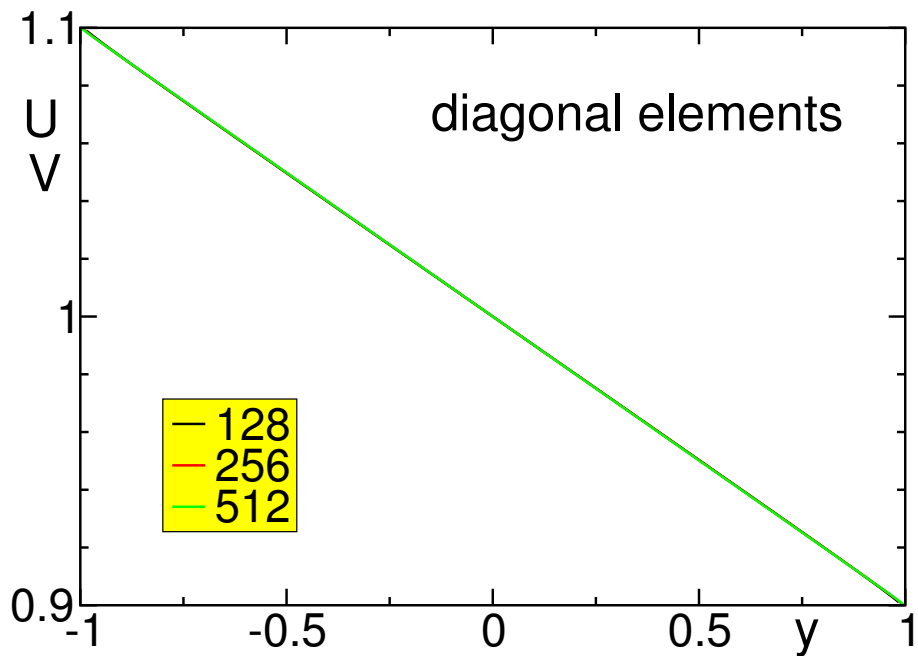
$$Q_i = aq_{i+1} - bq_i; \quad P_i = p_i$$

$$a = [\omega^2 + k/2 + (\omega^2 k + k^2/4)^{1/2}]^{1/2} \quad b = \omega^2/a.$$

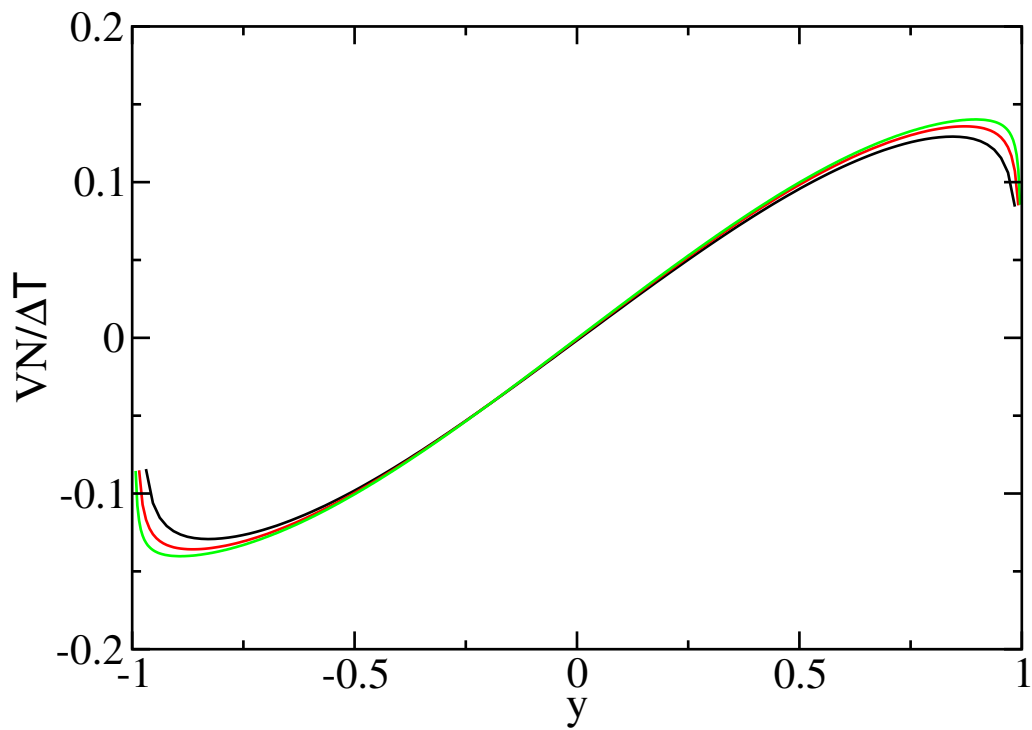
## NEW COVARIANCE MATRIX

$$\mathbf{C} = \begin{pmatrix} \mathbf{U} & \mathbf{Z} \\ \mathbf{Z}^\dagger & \mathbf{V} \end{pmatrix}$$

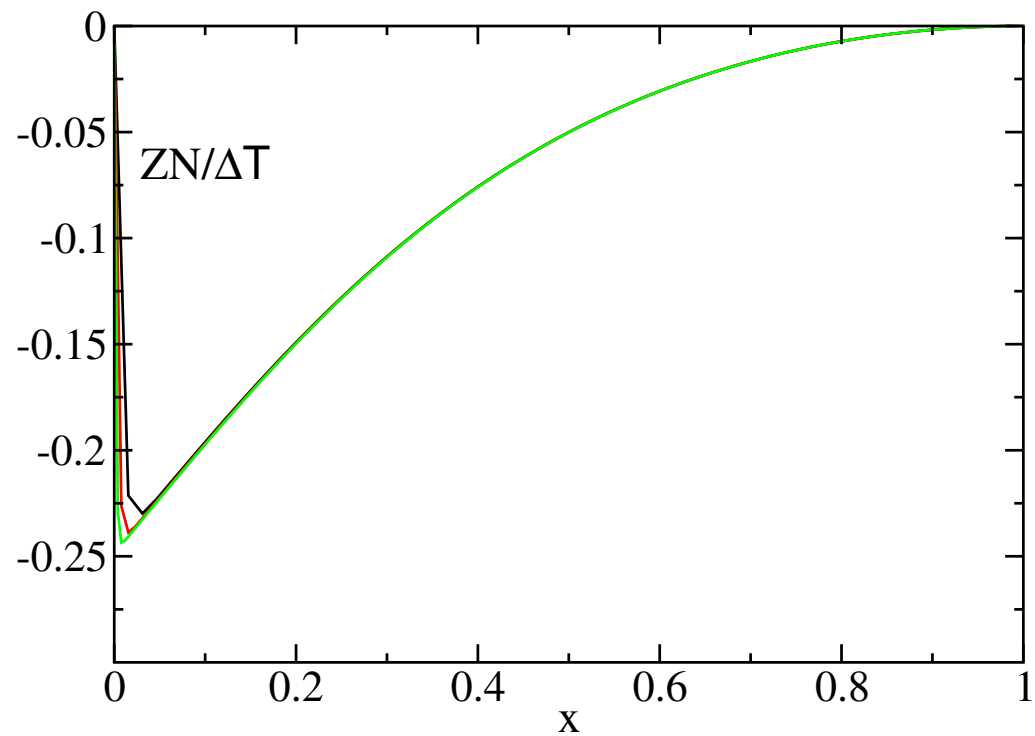
# WITH ON-SITE POTENTIAL



$$y = (i + j - N)/N \quad x = (i - j)/N$$



$$i - j = 1$$



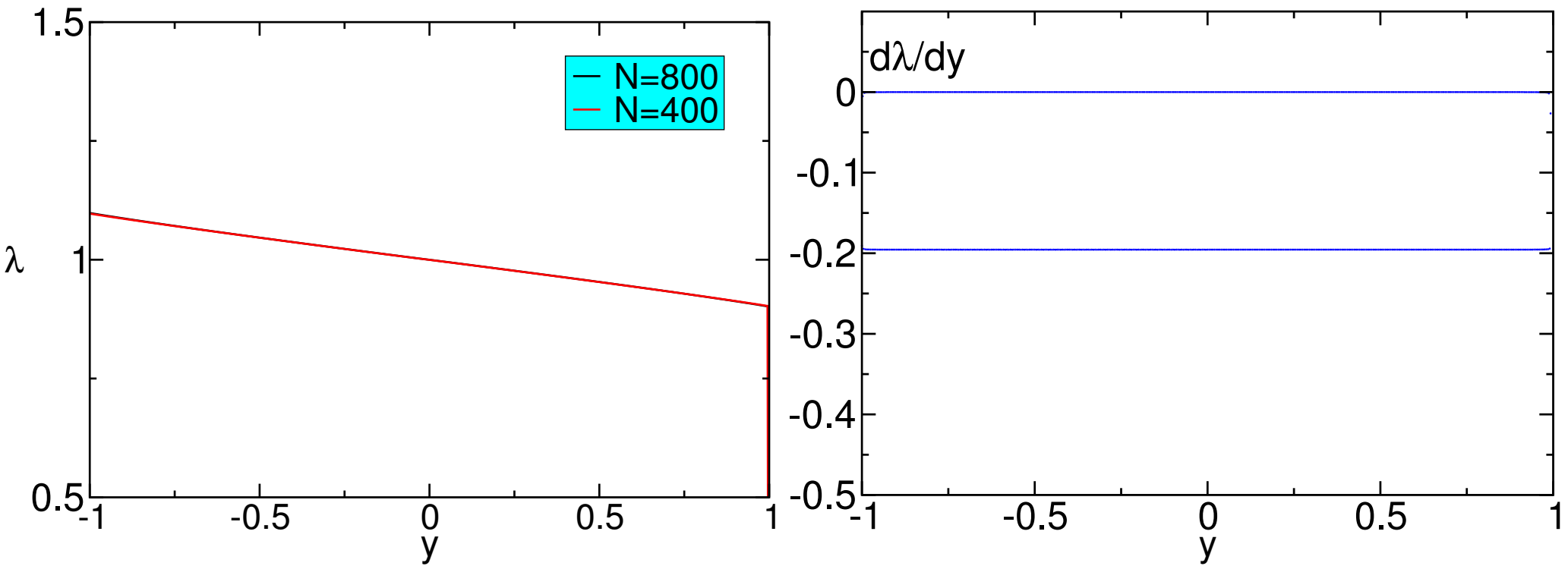
$$y = 0$$

antidiagonal

# PRINCIPAL COMPONENT ANALYSIS

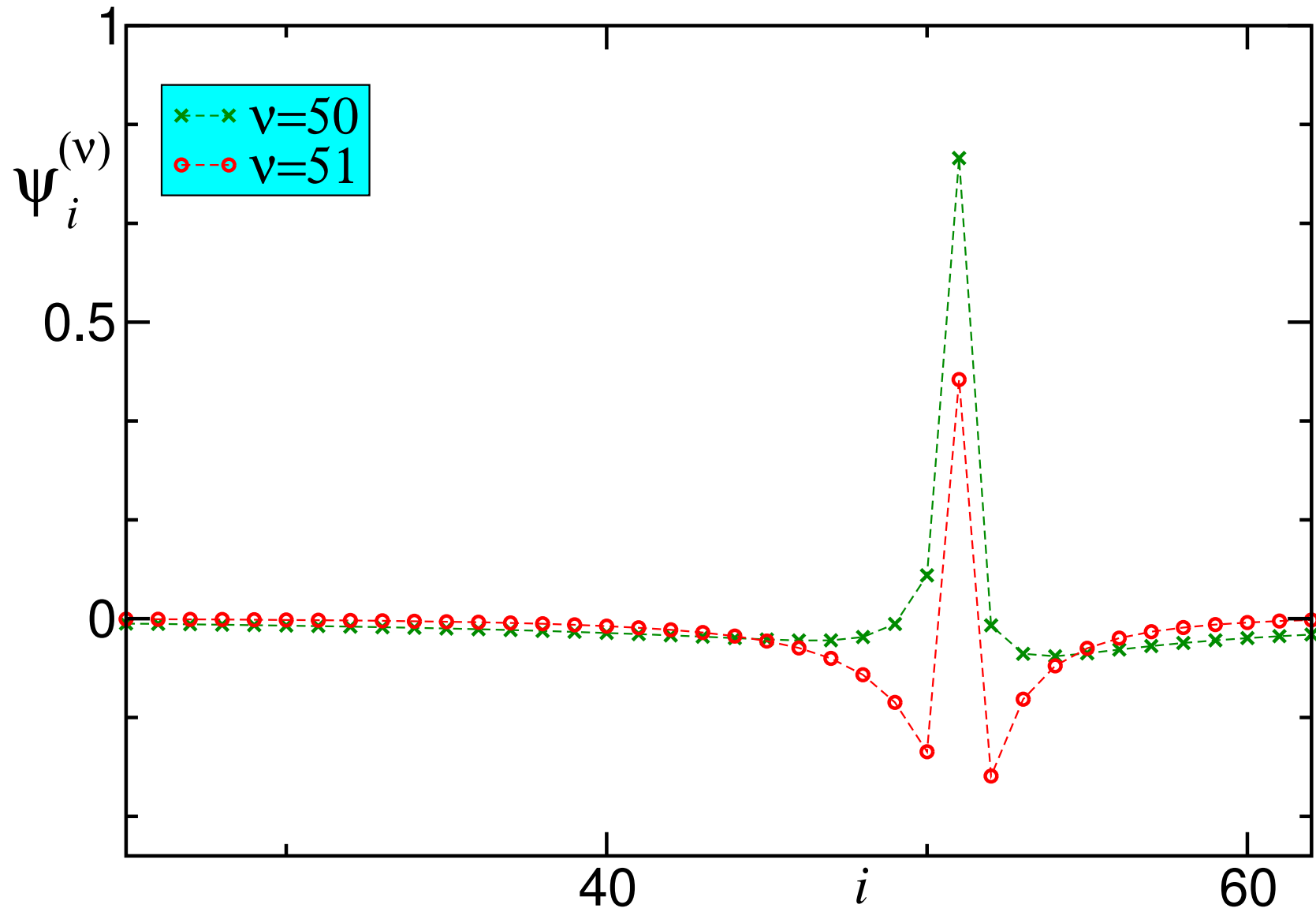
$$\mathbf{C}\psi^{(\nu)} = \lambda_{\nu}\psi^{(\nu)}, \quad \nu = 1, \dots, 2N$$

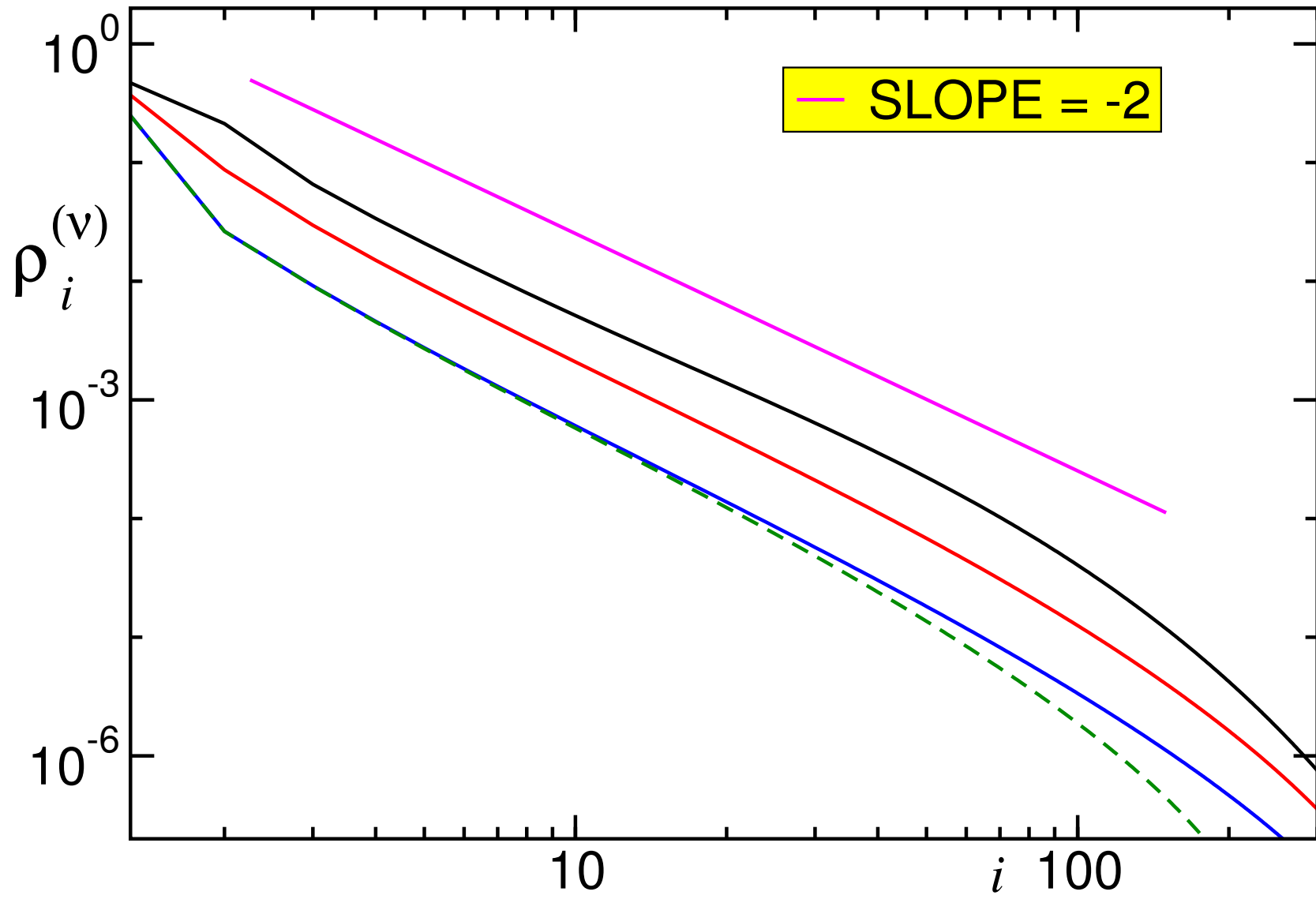
EIGENVALUES



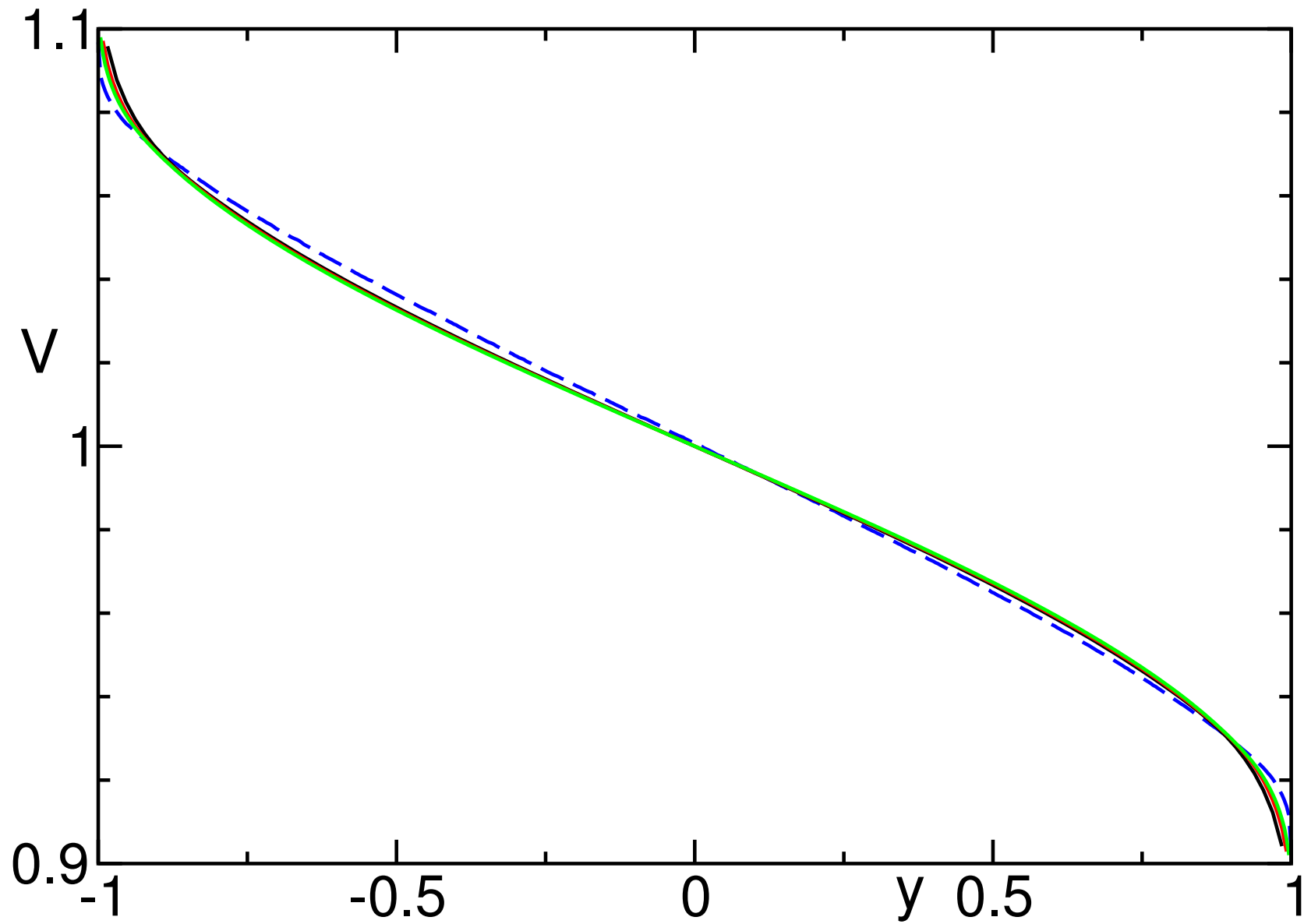


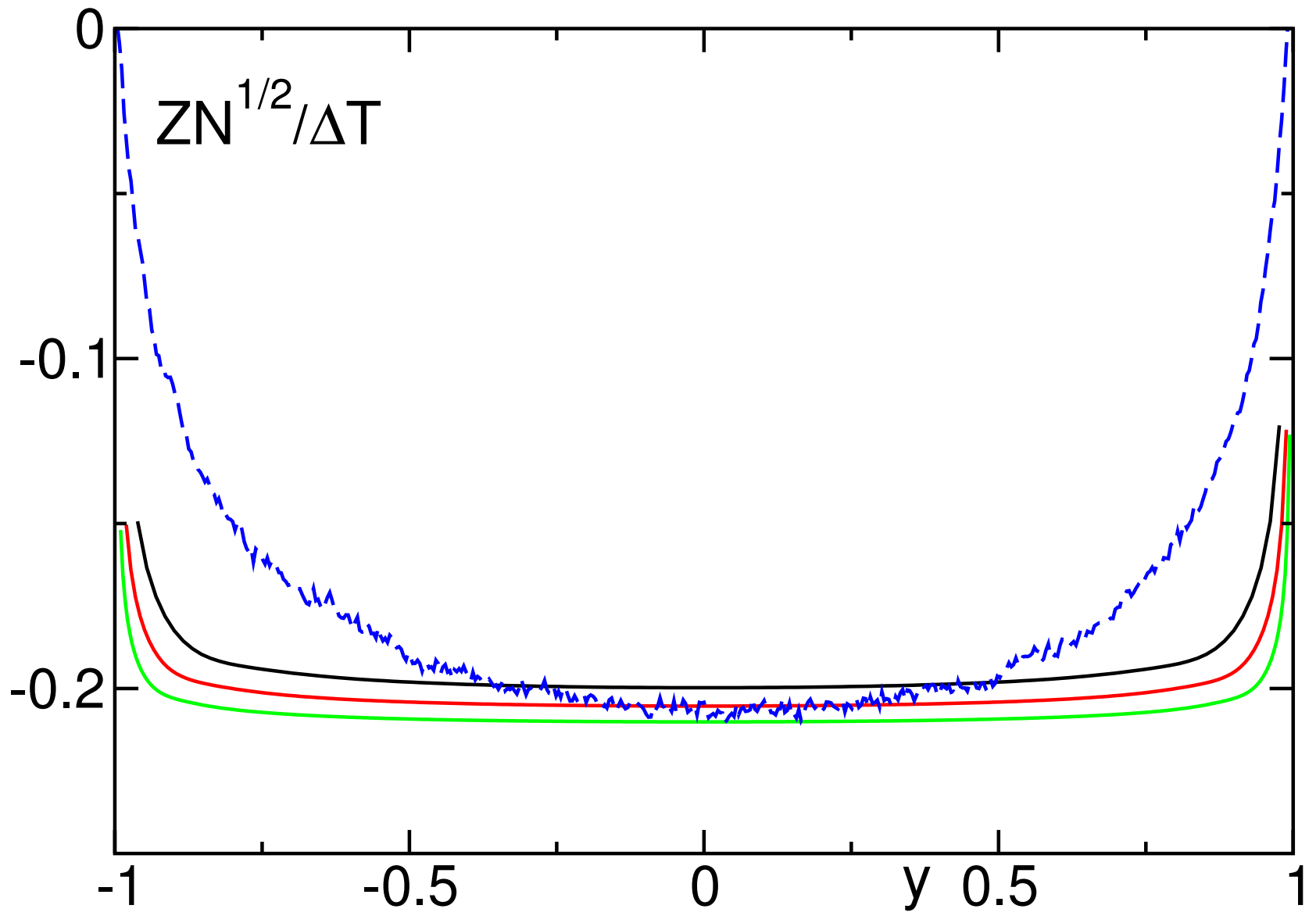
# EIGENVECTORS



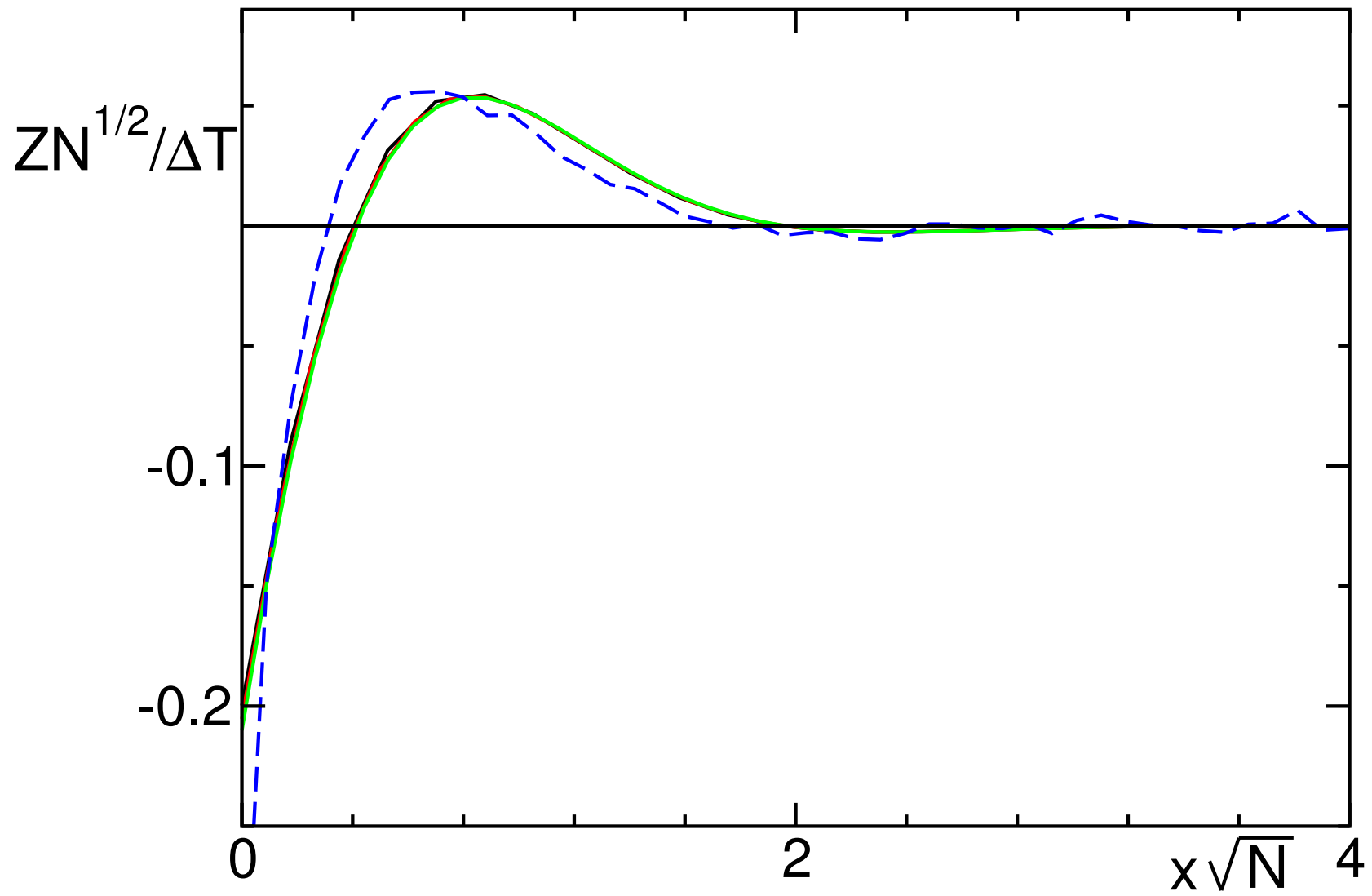


# WITH TRANSLATIONAL INVARIANCE





# ANTIDIAGONAL



# THE CONTINUUM LIMIT

with S. Lepri and C. Mejia-Monasterio

$$x = (i - j)\varepsilon \quad y = \frac{(i + j)\varepsilon^2 - 1}{|1 - (i - j)\varepsilon^2|}$$

where  $\varepsilon$  is a smallness parameter,

$$\varepsilon = \frac{1}{\sqrt{N}}$$

The leading component of the matrix  $Z$  satisfies the equation

$$4\omega^2 Z_{yy}^{(0)} = \gamma^2 Z_{xxxx}^{(0)}$$

