Nonlinear transport phenomena in low-dimensional lattices

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References

- R. Khomeriki, S. Lepri and S. Ruffo: *Pattern formation and localization in the forced-damped FPU lattice*, Phys. Rev. E, **64** 056606 (2001).
- R. Khomeriki, S. Lepri and S. Ruffo:*Excitation of travelling multibreathers in anharmonic chains*, Physica D, **168-169**, 152 (2002).
- Y.A. Kosevich, R. Khomeriki and S. Ruffo:*Supersonic discrete kink-solitons with “magic” wavenumber in anharmonic lattices*, Europhysics Letters, **66**, 21 (2004).
- R. Khomeriki, S. Lepri and S. Ruffo: *Nonlinear supratransmission in the FPU model*, Phys. Rev. E, **70** 066626 (2004).
- T. Dauxois, R. Khomeriki, F. Piazza and S. Ruffo: *The Anti - FPU problem*, CHAOS, **15**, 015110 (2005).
- G. Miloshevich, R. Khomeriki and S. Ruffo: *Stochastic resonances in the Fermi-Pasta-Ulam chain*, preprint (2008).

PLAN

- Homogeneous forcing
  - The forced-damped FPU lattice
  - Stability analysis of the zone boundary mode
  - Nonlinear waves, standing and travelling multibreathers
- Forcing from a boundary
  - Sharp pulse method: supersonic kinks
  - Supratransmission
- Stochastic forcing
  - Multiple solutions
  - Resonance

The forced-damped FPU lattice

Equations of motion

$$\ddot{u}_n = u_{n+1} + u_{n-1} - 2u_n + (u_{n+1} - u_n)^3 + (u_{n-1} - u_n)^3 \\ -\gamma \dot{u}_n + f\cos(\omega t - \pi n)$$

- u_n, displacement of the n-th oscillator with respect to its equilibrium position
- f forcing strength at variable driving frequency ω acting on the shortest wavelength (π mode)
- γ damping acting on all the modes

For vanishing damping and forcing:$\gamma = f = 0$ (Hamiltonian lattice limit), the π-mode is modulationally unstable above a critical energy density $\epsilon_c = E_c/N \approx \pi^2/(3N^2)$.

Its destabilization leads to the formation of a chaotic breather.

Mode-coupling

Creation-annihilation operators representation

$$u_n = \frac{1}{2} \sum_k \left[a_k \, e^{i(\omega t + kn)} + a^+_{-k} \, e^{-i(\omega t - kn)} \right] ,$$

Weak damping limit $\gamma \ll \omega$ and neglecting $\ddot{a}_k$

$$-2i\omega \dot{a}_k - i\omega\gamma a_k =$$

$$= (\omega_k^2 - \omega^2) a_k + \delta_{k,\pi} f + 6 \sum_{q_1,q_2} G^k_{q_1,q_2} a_{q_1} a_{q_2} a^+_{q_1+q_2-k} ,$$

with $G^k_{q_1,q_2}$ some calculable mode-coupling coefficients and $\omega_k^2 = 2(1 - \cos k)$.
All our results are based on these equations and consist in

- studying the dynamics of the complex amplitude a_π of the π-mode, when all the energy is put in the π-mode
- computing the critical forcing threshold f_c above which the π-mode is “modulationally" unstable and begins to exchange energy with other modes
- deriving the approximate analytical expression of the modulated standing wave arising at frequencies ω inside the “phonon band" $\omega < \omega_* \approx 2 + \sqrt{3}\gamma/2$

π-mode dynamics

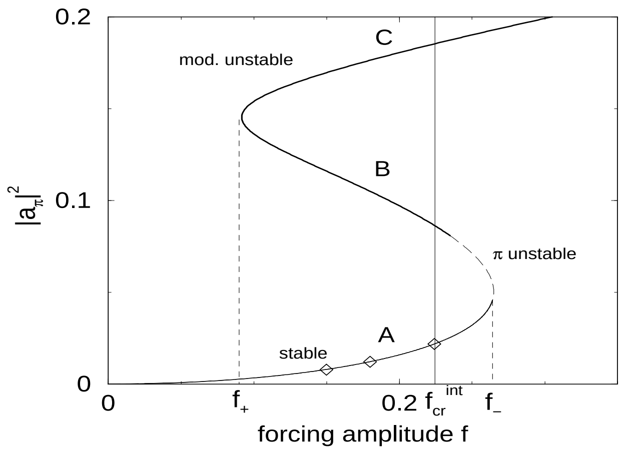
When the π-mode is stable, the dynamics of the complex amplitude a_π is given by

$$-2i\omega\dot{a}_\pi - i\omega\gamma a_\pi = (4-\omega^2)a_\pi + f + 12a_\pi|a_\pi|^2$$

Stationary solutions ($\dot{a}_\pi = 0$) solving

$$a_\pi = \frac{f}{\omega^2 - 4 - 12|a_\pi|^2 - i\gamma\omega} \ ,$$

both for the amplitude and for the phase of a_π.
-$\omega < \omega_*$ (in-band), only one real root
-$\omega > \omega_*$ (out-band), three roots in $[f_+, f_-]$



$\gamma = 0.1, \omega = 2.4$

Critical forcing

Linearizing around the π-mode solution

$$-2i\omega\dot{a}_k - i\omega\gamma a_k = (\tilde{\omega}_k^2 - \omega^2)a_k + 3\omega_k^2 a_\pi^2\, a_{-k}^+ \ ,$$

where $\tilde{\omega}_k^2 = (1 + 6|a_\pi|^2)\omega_k^2$.
-Look for solutions of the form $\exp(\nu_k t)$.
-For $\omega < \omega_*$ (in-band), full analytical solution

$$f_c = \sqrt{\frac{\gamma}{3(\omega - 2\gamma)} \left\{ \left[\omega^2 - \frac{4(\omega - \gamma)}{\omega - 2\gamma}\right]^2 + \gamma^2\omega^2 \right\}} \quad \text{critical forcing}$$

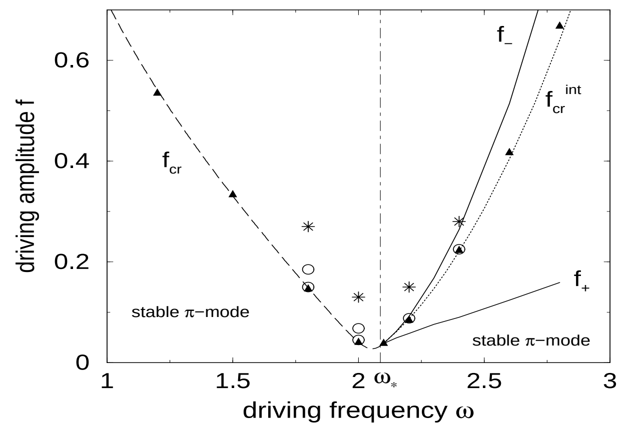
$$|a_\pi|_c^2 = \gamma/(3(\omega - 2\gamma)) \quad \text{critical amplitude}$$

$$\cos k_M = 1 - \omega^2/(2(1 + 6|a_\pi|_c^2)) \quad \text{fastest growing mode}$$

-For $\omega > \omega_*$ (out-band) a new type of instability appears, which destabilized the lower branch A at f_{cr}^{int} before f reaches the end point f_-.
Numerical determination of f_{cr}^{int}

Stability chart

Control parameter plane (ω, f) for $\gamma = 0.1$.



$\omega \ < \ \omega_*(in - band) \longrightarrow$ Nonlinear wave
$\omega \ > \ \omega_*(out - band) \longrightarrow$ Multibreather state
The dashed line for $\omega < \omega_*$ is the critical forcing f_{cr} at which "modulational instability" occurs. The region for $\omega > \omega_*$, where three solutions exist, is bounded by the solid lines $f_\pm$. The dotted line within corresponds to the instability threshold f_{cr}^{int} (evaluated numerically from the internal dynamics of the π-mode). The full triangles are the numerical estimates of f_{cr} for the full system. Open circles left (resp. right) of the $\omega = \omega_*$ line denote points where standing waves (resp. breathers) occur. The stars are some parameter values for which space-time chaotic patterns have been detected.

In-band nonlinear wave

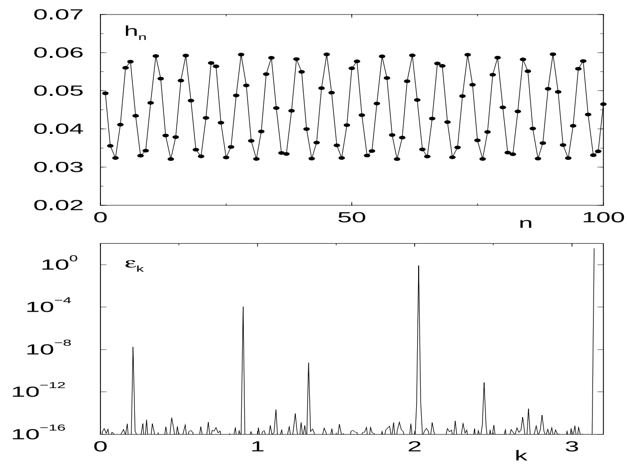
For $\omega < \omega_*$, stationary subdynamics in $(k_M, \pi) \longrightarrow$ Standing modulated waves.

$$u_n = |a_\pi|(-1)^n \cos(\omega t + \theta_\pi) + 2|a_{k_M}| \cos(k_M n) \cos(\omega t + \theta_{k_M})$$

Infinite numbers of harmonics $k_n = k_M n + (n-1)\pi$, $n = 1, \ldots$ (at variance with forced Klein-Gordon lattices).
($\omega = 1.8$, $f = 0.150$ and $\gamma = 0.1$)

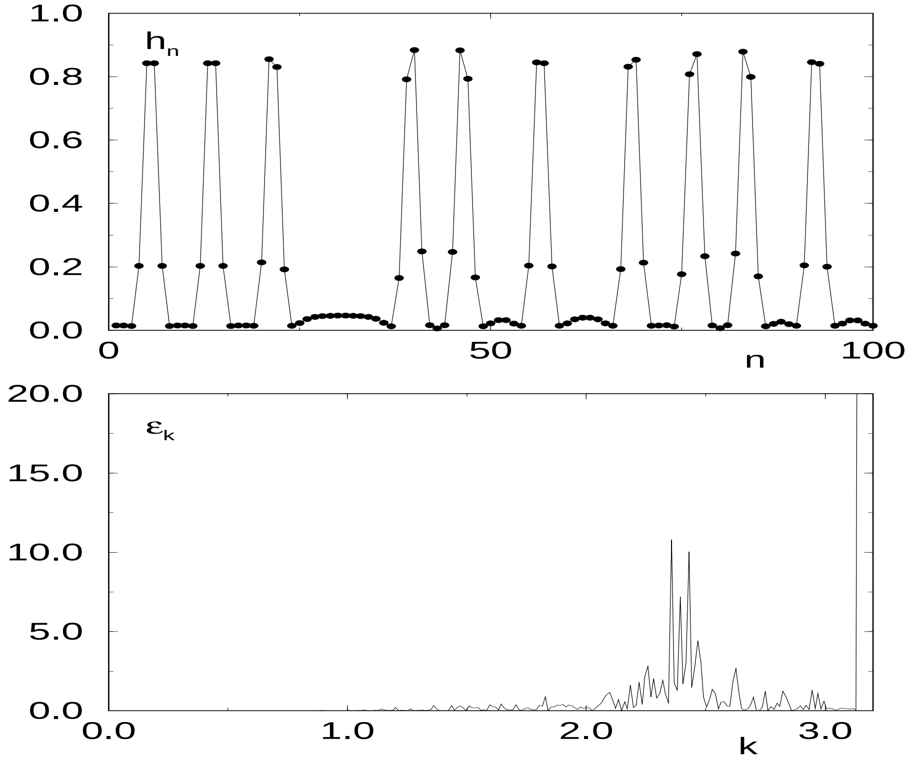
$$|a_\pi|^2 = 0.02085 \qquad |a_{k_M}|^2 = 4.88\,10^{-4}\ (numerical)$$
$$|a_\pi|^2 = 0.02085 \qquad |a_{k_M}|^2 = 4.27\,10^{-4}\ (theoretical)$$

Energy density h_n along the chain and spectrum of mode energies ϵ_k.



Out-band multibreather

For $\omega > \omega_*$ modulational instability leads to the formation of a **stable** unevenly spaced distribution of breathers, a sort of multibreather. In the figure below, $\omega = 2.4$, $f = 0.225$. The displacement pattern shows an in-phase breather over an anti-phase π-mode background.



Forcing the $p < \pi$ mode

Equations of motion

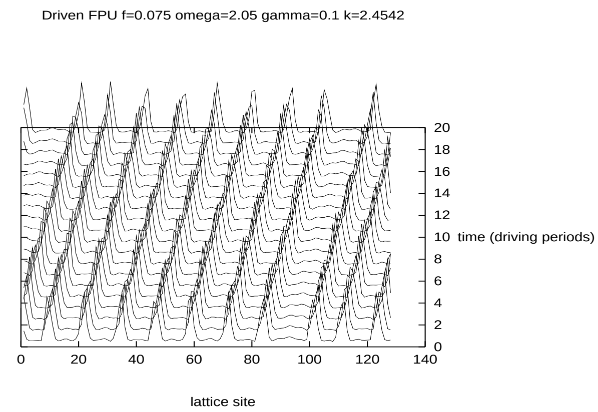
$$\ddot{u}_n = u_{n+1} + u_{n-1} - 2u_n + (u_{n+1} - u_n)^3 + (u_{n-1} - u_n)^3 \\ -\gamma \dot{u}_n + f \cos(\omega t - pn)$$

Now $\pi/2 < |p| < \pi$.

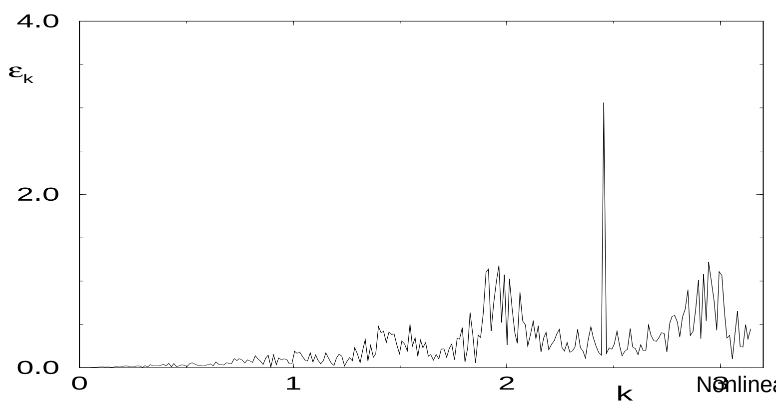
- For $\omega < \omega_* \approx \omega_p + \sqrt{3}/(2\gamma)$ (in-band), p-mode destabilization leads to the formation of a travelling modulated wave, well described by a triplet of modes $k_*, p, 2p - k_*$.
- For $\omega > \omega_*$, the destabilization leads to a travelling multibreather, which, in the low-amplitude limit, can be described by appropriate solutions of a driven-damped nonlinear Schrödinger equation.

Travelling multibreather

Travelling multibreather pattern generated after the modulational instability of the p-mode for a lattice of $N = 512$ sites. Here, $\omega = 2.05$, $f = 0.075$, $p = 2.4542$.



Spatial spectrum of the travelling multibreather. $\epsilon_k = |\dot{U}_k| + \omega^2 |U_k|$, where U_k is the k-th component of the Fourier spectrum of the displacement field u_n.



Sharp pulse

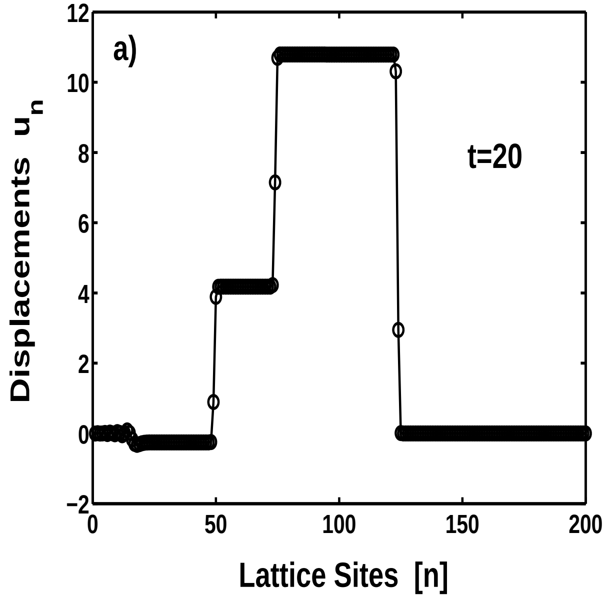
FPU

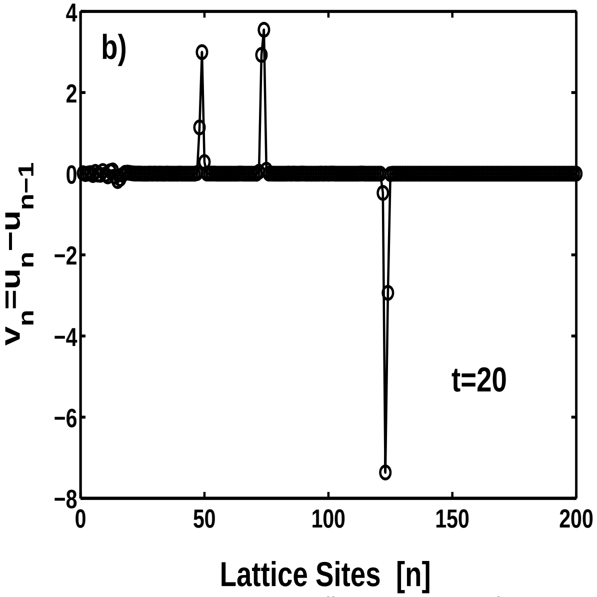
$$\ddot{u}_n = u_{n+1} + u_{n-1} - 2u_n + (u_{n+1} - u_n)^3 + (u_{n-1} - u_n)^3,$$

$$u_1 = A_0 \cdot sin\Big[\omega_0(t_0 - t)\Big] \quad \text{if} \quad 0 < t \le t_0 \quad \text{and}$$

$$u_1 = 0 \quad \text{if} \quad t > t_0,$$

The right end u_N is pinned, $v_n = u_{n+1} - u_n$ (relative displacement)
$A_0 = 11, \omega_0 = \sqrt{2}, t_0 = 1, N = 200$





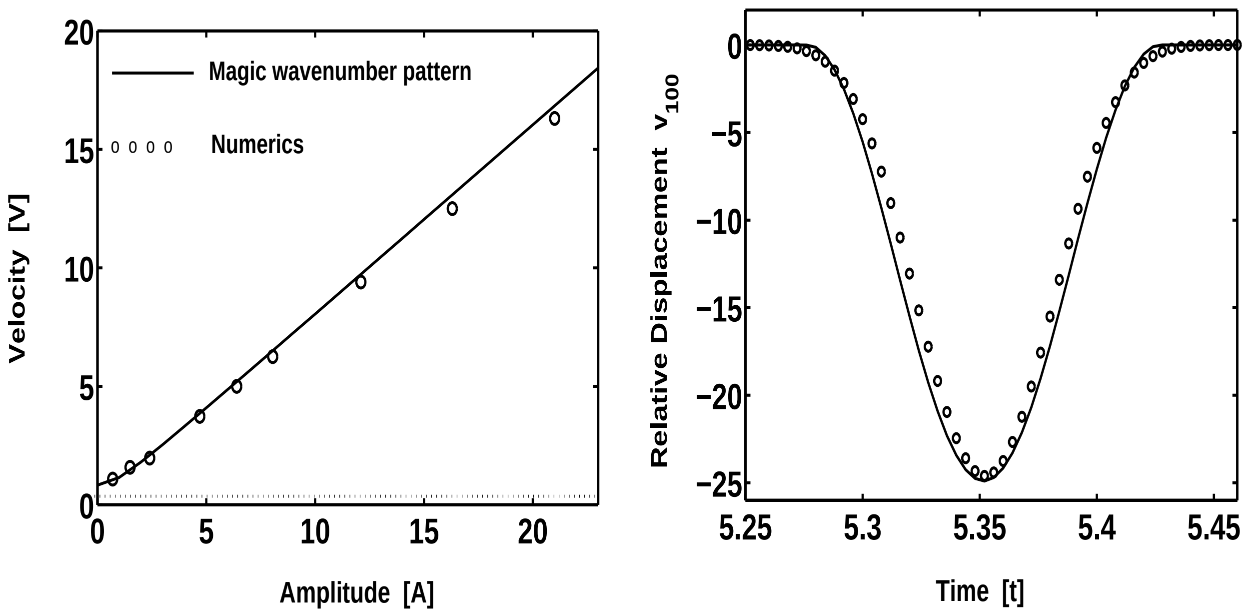
Approximate solution

Truncated magic wavenumber $\frac{2\pi}{3}$ nonlinear travelling wave

$$v_n = \pm\frac{A}{2}\left[1+\cos\left(\frac{2\pi}{3}n-\omega t\right)\right] \quad \text{if} \quad -\pi < \frac{2\pi}{3}n-\omega t < \pi$$

Frequency ω and velocity V in rotating wave approximation

$$\omega = \sqrt{3+(45/16)A^2}; \qquad V=\omega/(2\pi/3) = 3\sqrt{3+(45/16)A^2}/(2\pi).$$



Exponential tails

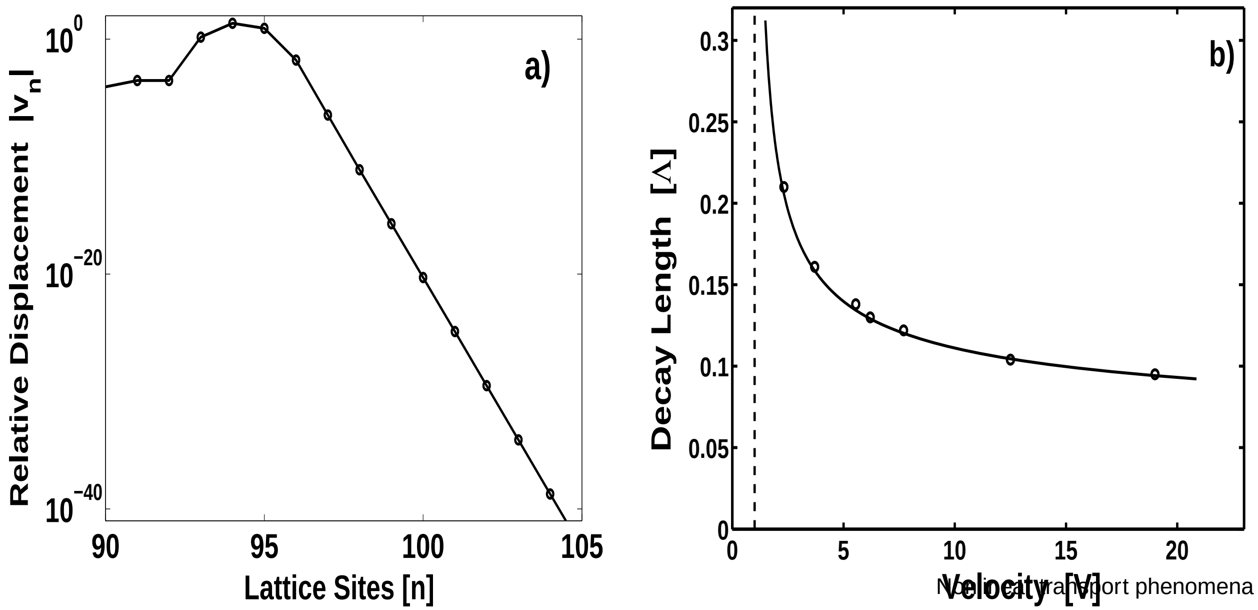
Kinks have a localized bulk sided by exponential tails. The decay length Λ is a function of velocity. This dependence can be derived analytically representing the relative displacements in the tails as

$$v_n \sim \exp\left[\pm(n - Vt)/\Lambda\right].$$

In the limit $n - Vt \gg \Lambda$, one gets

$$V^2 = 4\Lambda^2 \sinh^2(1/2\Lambda)$$

The decay length Λ is real positive only for supersonic excitations with $V > 1$ and it diverges for $V = 1$.



Mechanically excited FPU chain

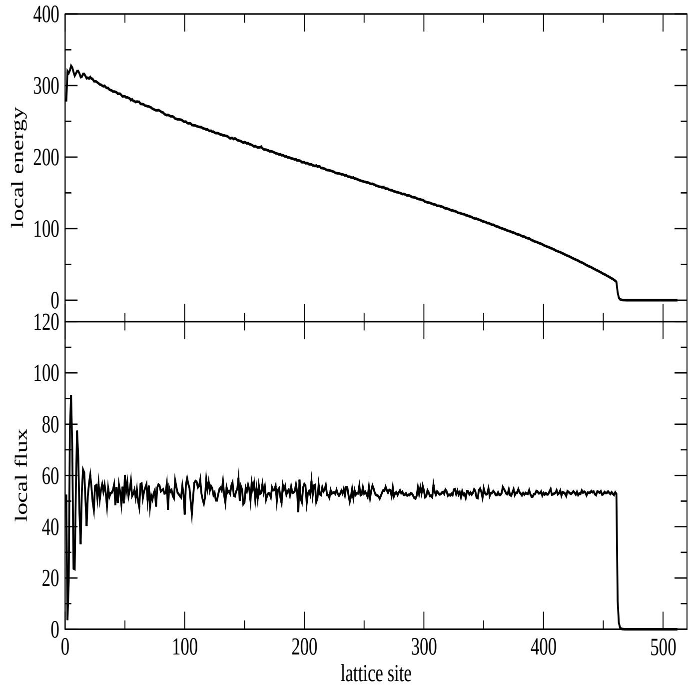
FPU bulk equations

$$\ddot{u}_n = u_{n+1} + u_{n-1} - 2u_n + (u_{n+1} - u_n)^3 + (u_{n-1} - u_n)^3$$

Left border driving $u_0(t) \;=\; A\cos\omega t$
Righ border damping Damping is applied to N_D rightmost sites (typically 10% of the total) by adding a viscous term $-\gamma\dot{u}_n$ to their equation of motion.
Average energy flux $j = \sum_n j_n/N$

$$j_n = \frac{1}{2}(\dot{u}_n + \dot{u}_{n+1})\left[u_{n+1} - u_n + (u_{n+1} - u_n)^3\right]$$

Stationary state



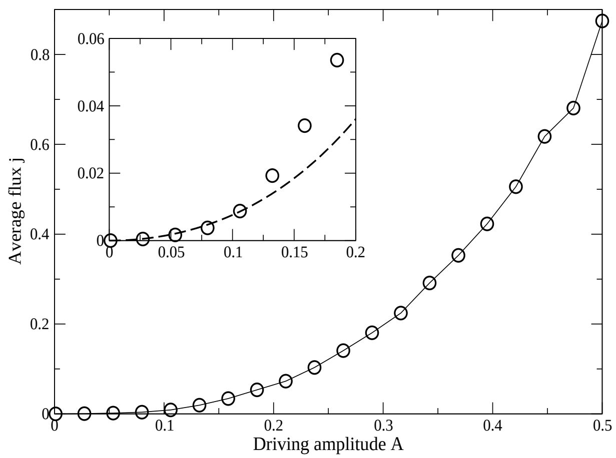
The local flux is computed for $\omega = 3.5, A = 1.27, \gamma = 5$.

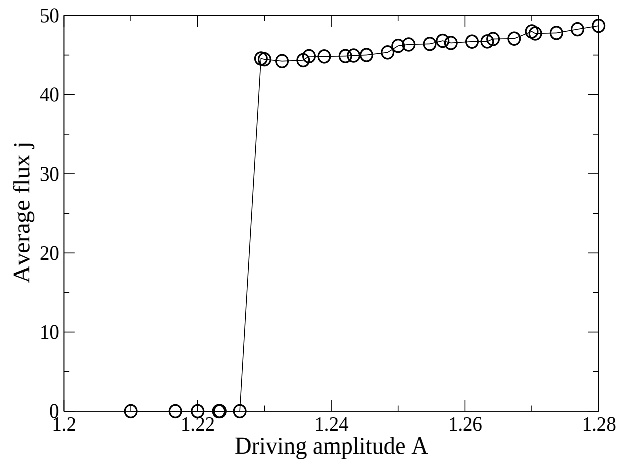
Supratransmission

In the quasi-harmonic approximation

$$j = \frac{1}{2} v(k, A)\, \omega^2 A^2 \ ,$$

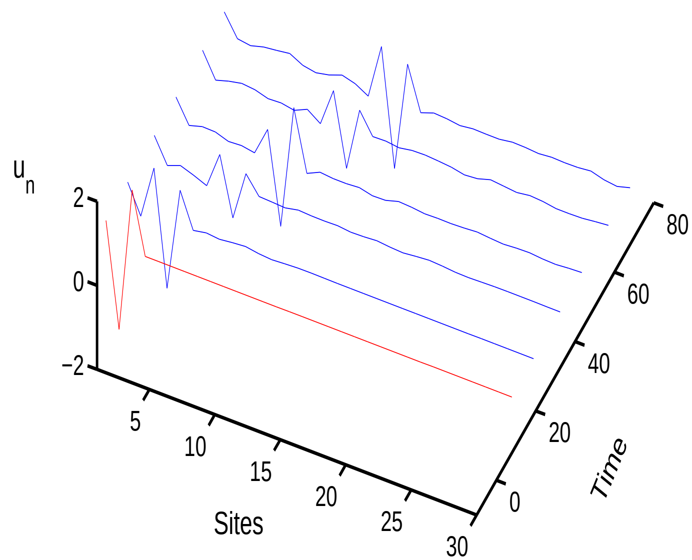
where $v(k, A)$ and ω are the group velocity and the frequency of nonlinear phonons, respectively.





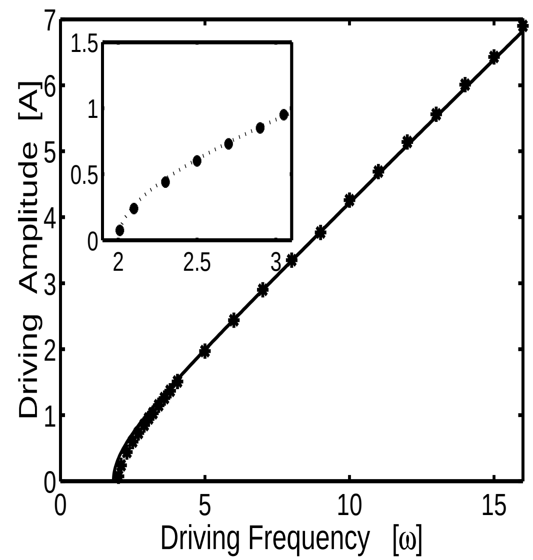
Left figure $\longrightarrow \omega = 1.8$,$\gamma = 5$ Right figure $\longrightarrow \omega = 3.5$,$\gamma = 5$

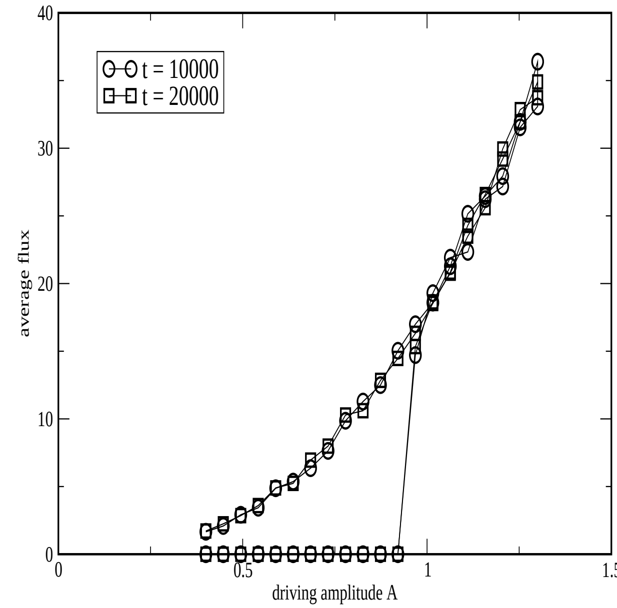
Boundary breather excitation



Driving frequency $\omega = 5.12$ and amplitude $A = 0.5$, below the supratransmission threshold $A_{th} = 2.05$.
Above the threshold a repeated excitation of breathers from the boundary occurs.

Transition and hysteresis





Left graph: Comparison between analytic estimates and numerical values of threshold amplitudes vs. the driving frequency.
Right graph: Histeresis loop is for $\omega = 3, \gamma = 5, N = 512$.

Stochastic forcing

$$
\begin{aligned}
\ddot{u}_n &= u_{n+1} + u_{n-1} - 2u_n + (u_{n+1} - u_n)^3 \\
&\quad + (u_{n-1} - u_n)^3 - \gamma \dot{u}_n + \xi_n(t),
\end{aligned}
$$

Gaussian space uncorrelated white noise

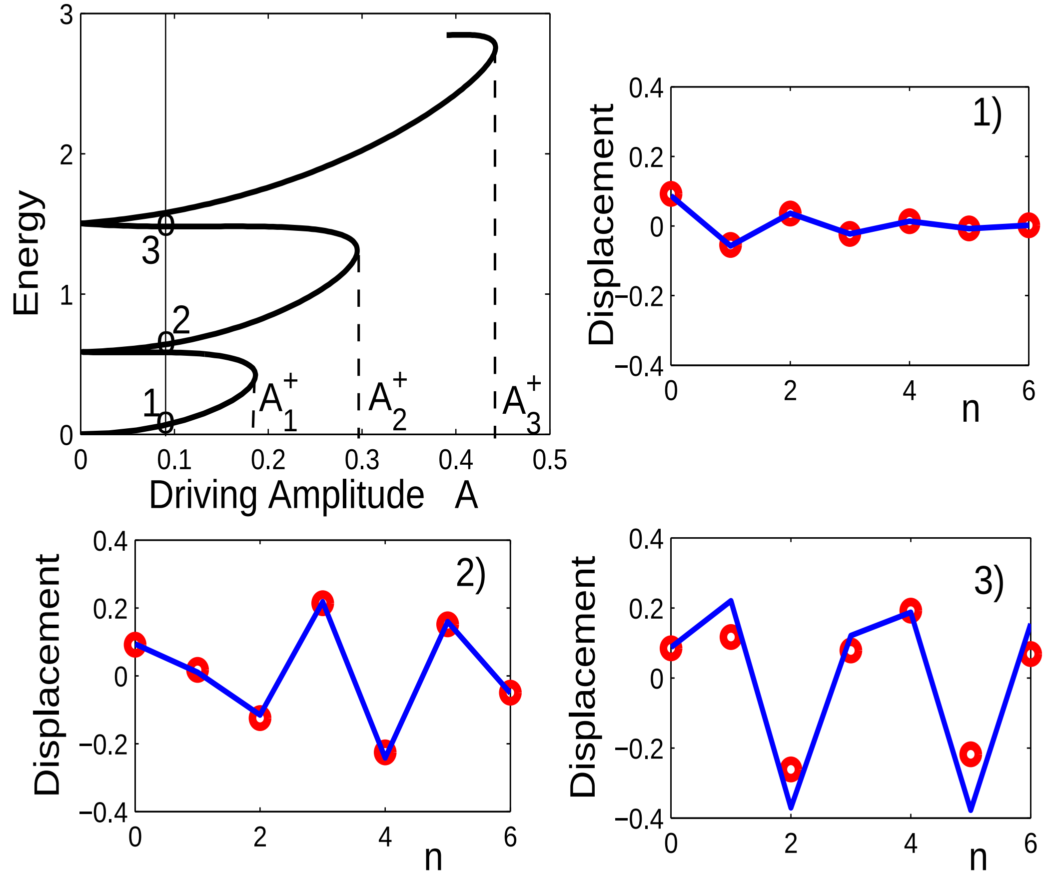
$$\langle \xi_m(t) \xi_n(0) \rangle = 2D\delta(t)\delta_{mn}$$

Boundary conditions

$$u_0(t) = A\cos(Wt)\cos(\Omega t) \qquad u_N(t) = u_{N+1}(t),$$

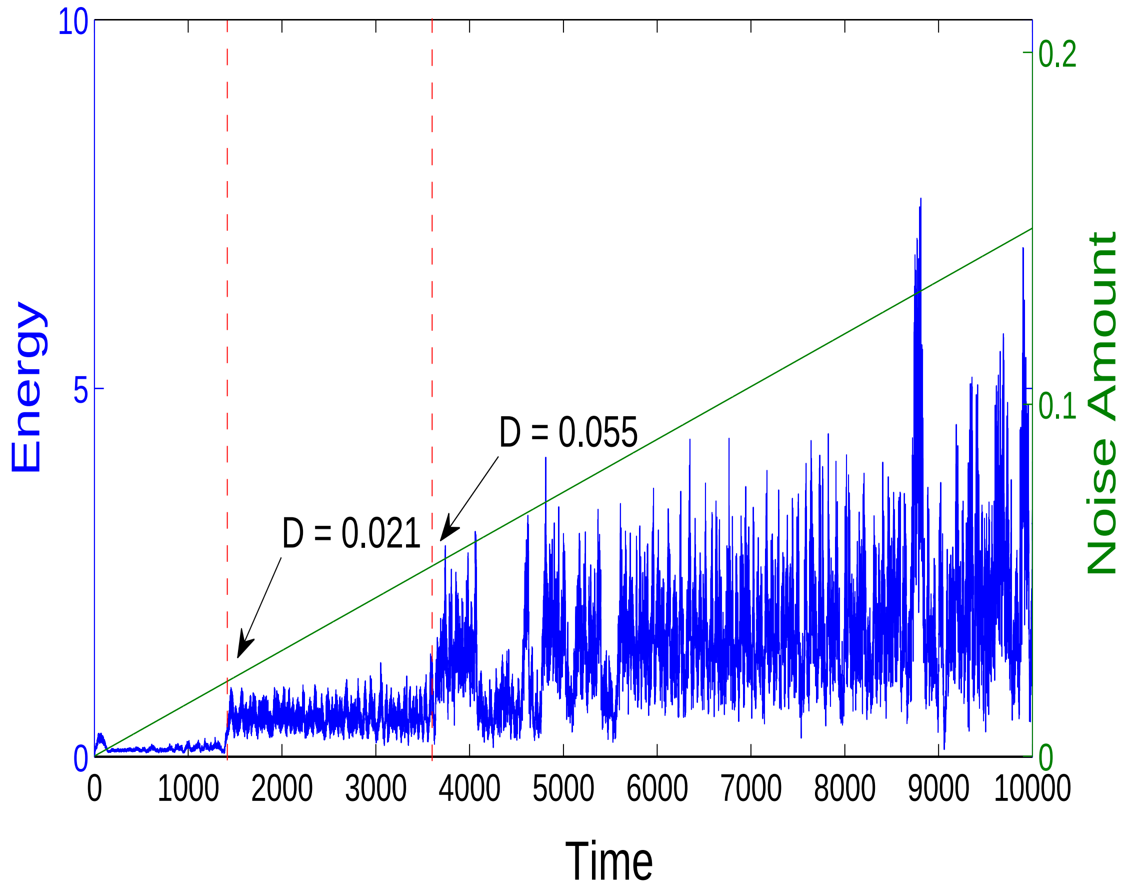
A is time modulated with frequency $W \ll \Omega \sim 2$

Solutions without damping and noise

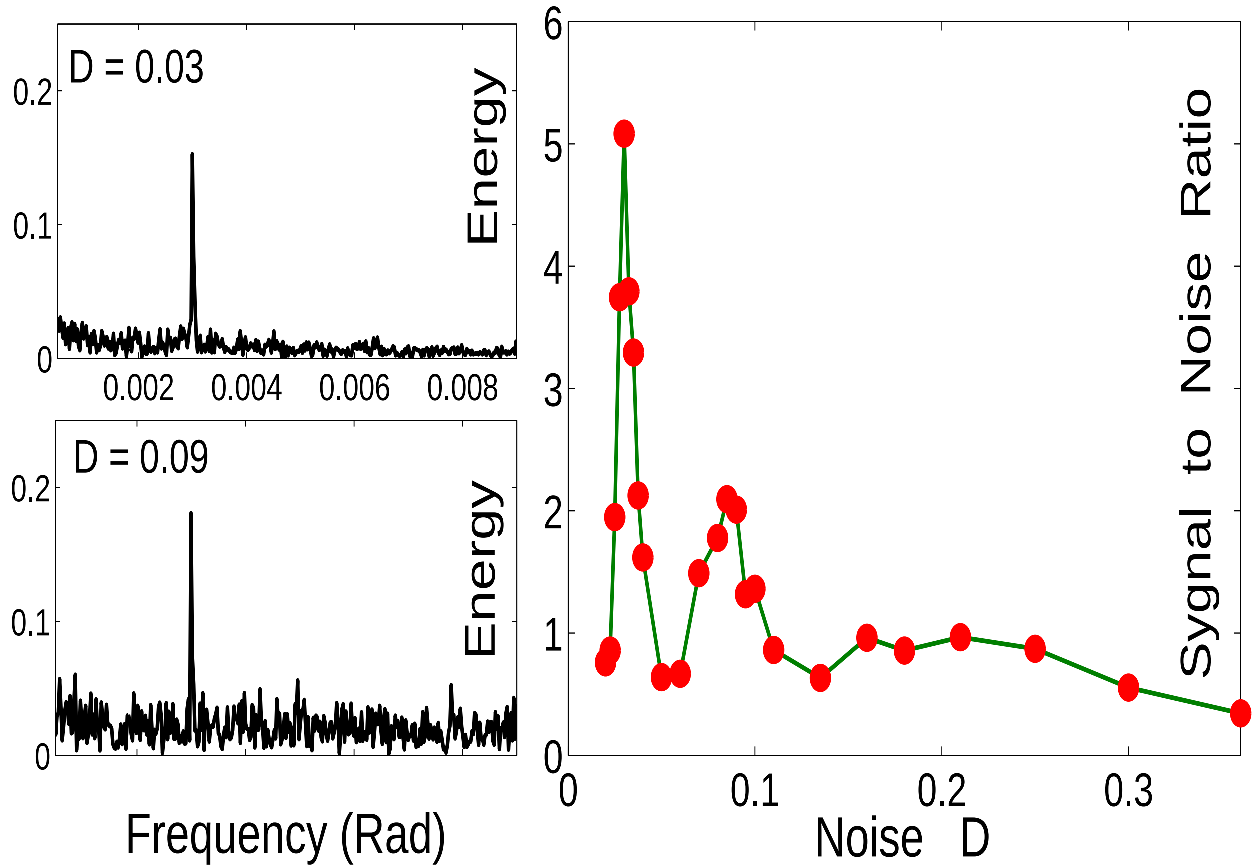


Forcing energy levels with noise

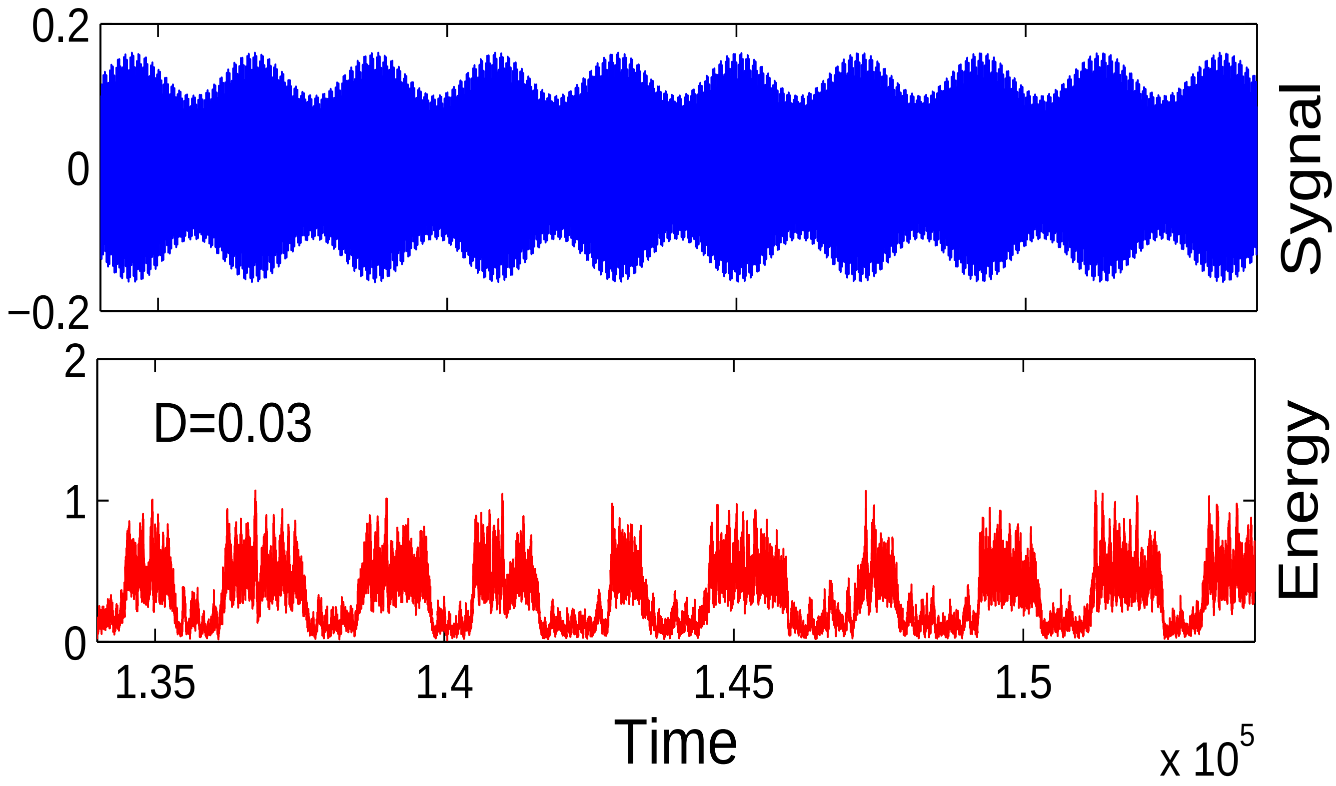
$W = 0$, no modulation



Multiple resonance-I



Multiple resonance-II



Applications to optical systems

- R. Khomeriki and S. Ruffo: *Landau-Zener tunnelling in waveguide arrays*, Phys. Rev. Lett.,**94**, 113904 (2005).
- R. Khomeriki, J. Leon, S. Ruffo:*Coexistence of Josephson oscillations and novel self-trapping regime in optical waveguide arrays*, Phys. Rev. Lett. **97**, 143902 (2006).
- R. Khomeriki, S. Ruffo and S. Wimberger: *Driven collective quantum tunneling of ultracold atoms in engineered optical lattices*, Europhysics Letters, **77**, 40005 (2007).
- R. Khomeriki, J. Leon, S. Ruffo and S. Wimberger: *Nonlinear dynamics in double square well potentials*, Theoretical and Mathematical Physics, **152**, 1122 (2007).

Reviews on FPU

- T. Dauxois, M. Peyrard and S. Ruffo: *The Fermi-Pasta-Ulam ”numerical experiment”: history and pedagogical perspectives*, Eur. J. Phys., **26**, S3-S11 (2005).
- A. J. Lichtenberg, R. Livi, M. Pettini and S. Ruffo: "Dynamics of oscillator chains", in *The Fermi-Pasta-Ulam problem*, G. Gallavotti Ed., Lecture Notes in Physics **728**, 21 (2008).
- T. Dauxois and S. Ruffo, *Fermi-Pasta-Ulam nonlinear lattice oscillations*, Scholarpedia, (2008).