Critical phenomena and renormalization-group flow of quantum field theories

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ABSTRACT: In the framework of the **renormalization-group (RG) theory**, several **critical phenomena** can be investigated by studying the **RG flow of an effective Landau-Ginzburg-Wilson (LGW) Φ^4 theory**, having an N-component order parameter as fundamental field, and containing up to 4th-order polynomials of the field. I discuss the **general properties of the RG flow of Φ^4 theories**, and present an **overview of RG field-theory results for physically interesting LGW Φ^4 theories**, whose results apply to liquids, magnets, disordered and/or frustrated spin systems, to the finite-T transition in hadronic matter, etc...

Critical phenomena are observed in many physical systems

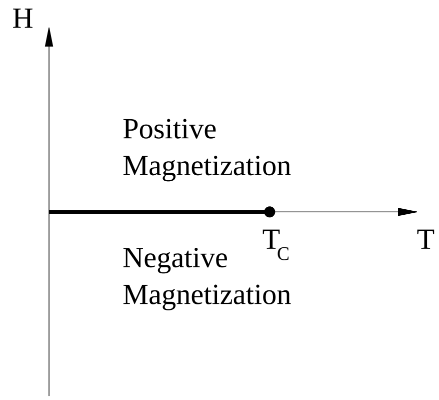
There are two broad classes of phase transitions:

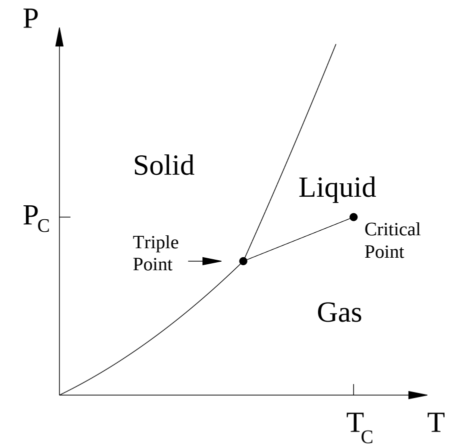
first order → discontinuity in thermodynamic quantities

continuous → nonanalytic behavior due to a diverging length

Examples of **continuous transitions**:

- magnetic transitions
- liquid-vapor in fluids





• first general framework was proposed by Landau (1937), based on an expansion of the free energy in powers of the order parameter, corresponding to mean-field approximation

• Renormalization-group (RG) theory by Wilson (1971)

In the framework of the RG theory, several critical phenomena can be investigated by studying the RG flow of Φ^4 theories with an N-component fundamental field Φ, and containing up to 4th-order polynomials of the field.

O(N)-symmetric models $\rightarrow \mathcal{L} = (\partial_\mu \vec{\Phi})^2 + r\vec{\Phi}^{\,2} + u\,(\vec{\Phi}^{\,2})^2$,

but also more complicated **multi-parameter Φ^4 theories** with several quadratic and quartic parameters, depending on the nature of the order parameter and the symmetry-breaking pattern

$$\mathcal{L} = \sum_i [(\partial_\mu \Phi_i)^2 + r_i \Phi_i^2] + \sum_{ijkl} u_{ijkl}\, \Phi_i \Phi_j \Phi_k \Phi_l$$

Results for their RG flow apply to several physical systems, such as liquids, magnets, the finite-T transition in hadronic matter, etc...

Plan of the talk

- RG theory of critical phenomena and universality
- Field-theory approach based on LGW Φ^4 theories
- RG flow of multiparameter Φ^4 theories, η conjecture
- Overview of results for some physically interesting Φ^4 theories:

O(N) models (ex.: polymers, liquid-vapor transition, superfluid transition in ^{4}He, magnets, etc...) but also more complicated Φ^4 theories with several quadratic and quartic parameters (ex.: frustrated spin models with noncollinear order, superfluid transition in ^{3}He, disordered spin models, magnets with impurities, multicritical behaviors in high-T_c superconductors, anisotropic antiferromagnets, quantum transitions in cuprates, finite-T transition in hadronic matter, etc...)

Continuous transitions are characterized by power-law behaviors

- Disordered (symmetric) phase ($t \equiv T/T_c - 1 > 0$, $h = 0$):

$$\xi \sim t^{-\nu}, \quad C_H \sim t^{-\alpha}, \quad \chi \sim t^{-\gamma}, \quad \chi \sim \xi^{2-\eta}$$

- Ordered (broken) phase ($t < 0$, $h = 0^+$): $C_H \sim |t|^{-\alpha}$, $M \sim |t|^{\beta}$
- Critical isotherm ($t = 0$, $h > 0$): $\chi \sim |h|^{-\gamma/\beta\delta}$, $\widetilde{G}(q) \sim q^{-2+\eta}$
- Scaling equation of state: $h = t^{\beta\delta} F(z)$, $z = M t^{-\beta}$
- Finite-size scaling, ex. $\chi \sim L^{2-\eta}$ at $t = 0$
- There are also critical behaviors characterized by exponential approaches: **LATTICE QCD** where $\xi \sim \exp(c\beta)$, and also 2D σ models, 2D KT transition

Main ideas to describe the critical behavior at a continuous transition

- **Order parameter** which effectively describes the critical modes

- **Scaling hypothesis**: singularities arise from the long-range correlations of the order parameter, diverging length scale

- **Universality**: the critical behavior is essentially determined by a few global properties: the space dimensionality, the nature and the symmetry of the order parameter, the symmetry breaking

RENORMALIZATION-GROUP THEORY

- RG flow in a Hamiltonian space

- the critical behavior is associated with a fixed point of the RG flow

- only a few perturbations are relevant, the corresponding positive eigenvalues are related to the critical exponents ν, η, etc...

The Gibbs free energy obeys a scaling law

$$\mathcal{F}_{\rm sing}(u_1, u_2, \ldots, u_k, \ldots) = b^{-d}\mathcal{F}_{\rm sing}(b^{y_1}u_1, b^{y_2}u_2, \ldots, b^{y_k}u_k, \ldots)$$

u_k are nonlinear scaling fields (analytic functions of the model parameters)

In a standard continuous transition: **two relevant scaling fields** $u_t \sim t = T/T_c - 1$ (with $y_t = 1/\nu$) and $u_h \sim h$ (external field, with $y_h = (\beta + \gamma)/\nu$), and irrelevant u_i $(i \geq 3)$ with $y_i < 0$.

The critical point is approached by $u_t, t \longrightarrow 0$ and $u_h, h \longrightarrow 0$.

Setting $b^{y_t}|u_t| = 1 \rightarrow \mathcal{F}_{\rm sing} = |u_t|^{d/y_t}\mathcal{F}_{\rm sing}(u_h|u_t|^{-y_h/y_t}, u_i|u_t|^{-y_i/y_t})$

Since $u_i|u_t|^{-y_i/y_1} \longrightarrow 0$ $(i \geq 3)$, and introducing the length scale ξ,

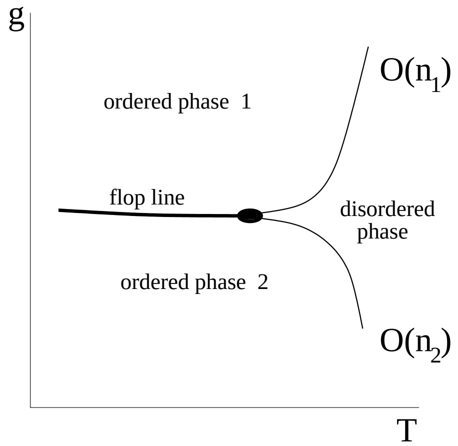
$$\mathcal{F}_{\rm sing} \approx \xi^{-d}\left[f(h\xi^{y_h}) + \xi^{-\omega}f_\omega(h\xi^{y_h}) + \ldots\right], \qquad \xi \sim t^{-\nu}$$

$O(\xi^{-\omega})$ arises from the leading irrelevant u_3, and $\omega = -y_3$.

Finite size: $\mathcal{F}_{\rm sing}(u_t, u_h, u_i) = L^{-d}\mathcal{F}_{\rm sing}(L^{y_t}u_t, L^{y_h}u_h, L^{y_i}u_i)$

The presence of other relevant perturbations beside t and h gives rise to **multicritical behaviors**. Let us set the external field $h = 0$.

At a multicritical point (for example where critical lines with different order parameter meet) other even relevant scaling fields beside T must be tuned to observe the multicritical scaling.



If $u_t \sim t$ and g_i are n further relevant scaling fields

$$\mathcal{F}_{\rm sing} \approx t^{d\nu} f(g_1 t^{-\phi_1}, \ldots, g_n t^{-\phi_n})$$

where $t = (T - T_{\rm mc})/T_{\rm mc}$, and $\phi_i = y_i/y_t > 0$.

When $n = 1$: $\mathcal{F}_{\rm sing} \approx t^{d\nu} f(g t^{-\phi})$, where g is the other relevant parameter beside the temperature, and $\phi > 0$ is the crossover exponent.

Scaling implies $T_c(g) = T_{\rm mc} + c g^{1/\phi}$

From the statistical model to the Φ^4 QFT

Ex.: **Ising model** defined on a d-dimensional lattice,

$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j, \quad \sigma_i = \pm 1, \qquad Z = \sum_{\{\sigma_i\}} \exp(-H/T)$

- The critical behavior is due to the long-range modes, with $l \gg a$.

- As a result of a blocking procedure ($b \ll l$), preserving the symmetry, $H_{\varphi^4} = \sum_{x,\mu} (\varphi_{x+\mu} - \varphi_x)^2 + u \sum_x (\varphi_x^2 - v^2)^2$, with $\varphi \in \Re$

- The limit $a \to 0$ of H_{φ^4} should not change the universality class $\mathcal{H}(\varphi) = \int d^d x \left[(\partial_\mu \varphi)^2 + r\varphi^2 + u\varphi^4 \right]$, $r - r_c \propto T - T_c$

- $Z = \int [d\varphi] \exp[-\mathcal{H}(\varphi)] \to$ QFT with $\mathcal{H}(\varphi) \to \mathcal{L}(\varphi)$

- RG flow by a set of RG equations for the correlation functions

The way back provides a nonperturbative formulation of an Euclidean QFT, from the critical behavior of a statistical model.

The RG theory provides the basis for the field-theory approaches.

Many critical phenomena can be described by LGW Φ^4 theories

$$\mathcal{L} = \sum_i [(\partial_\mu \Phi_i)^2 + r_i \Phi_i^2] + \sum_{ijkl} u_{ijkl}\, \Phi_i \Phi_j \Phi_k \Phi_l$$

where Φ is a N-component field. They are constructed by requiring a few global properties of the system, keeping terms up to 4th order.

UNIVERSALITY CLASSES within which the critical behavior is universal: • spatial dimension • nature of the critical modes and order parameter • symmetry and symmetry-breaking pattern

Ex: SUPERFLUID transition in ^{4}He along the λ-line: D=3, quantum amplitude of helium atoms as order parameter, U(1) symmetry

3-D XY UNIVERSALITY CLASS: $\mathcal{L} = |\partial_\mu \varphi|^2 + r\,|\varphi|^2 + u\,|\varphi|^4$ with a complex field φ, characterized by the critical exponents:
$\alpha = -0.0151(3)$, $\nu = 0.6717(1)$, $\eta = 0.0381(2)$

Perturbative schemes in field-theory approach

We are interested in the critical behavior of the "bare" correlation functions $\Gamma_n(p;r,u,\Lambda)$ of the ϕ^4 theory $\mathcal{L} = (\partial_\mu \vec{\varphi})^2 + r\vec{\varphi}^{\,2} + u\,(\vec{\varphi}^{\,2})^2$

• Massive zero-momentum scheme defined in the disordered phase

$$\Gamma_2(p) = Z_\varphi^{-1}[m^2 + p^2 + O(p^4)], \quad \Gamma_4(0) = Z_\varphi^{-2} m^{4-d} g, \quad \Gamma_{2,1}(0) = Z_t^{-1}$$

which relate the renormalized quantities m, g to the bare ones r, u.

• The critical limit $m \to 0$ can be studied by Callan-Symanzik RG equations for $\Gamma_n^{(r)}(p;m,g)$

$$\left[m\frac{\partial}{\partial m} + \beta(g)\frac{\partial}{\partial g} - \frac{1}{2} n \eta_\varphi(g) \right] \Gamma_n^{(r)}(p) = [2 - \eta_\varphi(g)] m^2 \Gamma_{n,1}^{(r)}(p;0)$$

- The RG functions $\beta(g) = m\partial g/\partial m$ and $\eta_{\varphi,t}(g) = \partial \ln Z_{\varphi,t}/\partial \ln m$ can be computed as power series of g (computed up to six, seven loops by Nickel etal for O(N) models)

- when $m \to 0$ the coupling g is driven toward an infrared-stable fixed point, i.e. a zero g^* of the β-function $\beta(g) \approx -\omega(g^* - g)$

- Using the RG equations, $\eta = \eta_\varphi(g^*), \quad 1/\nu = 2 - \eta_\varphi(g^*) + \eta_t(g^*)$

- The perturbative FT expansions are asymptotic: $S(g) = \sum_n s_n g^n$, $s_n \sim n^b (-a)^n n!$, $a > 0$. They must be resummed before evaluating at g^*, exploiting Borel summability and knowledge of the large-order behavior by computing instanton semiclassical solutions, which provide important nonperturbative information

- Alternative $\overline{\text{MS}}$ renormalization scheme defined at $T = T_c$, $\epsilon \equiv 4 - d$ expansion, but also exp in the coupling setting $\epsilon = 1$

Many results for the **3D Ising universality class**
(liquid-vapor systems, fluid mixtures, uniaxial magnets)
corresponding to $\mathcal{L} = (\partial_\mu \varphi)^2 + r\varphi^2 + u\varphi^4$ with $\varphi \in \Re$

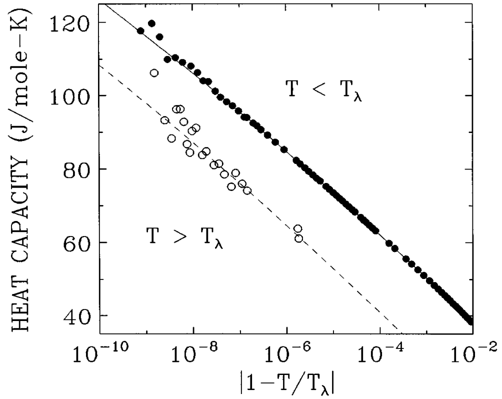
		ν	η	β
EXPT	liq-vap	$0.6297(4)^*$	0.042(6)	0.324(2)
	fluid mix	$0.6297(7)^*$	0.038(3)	0.327(3)
	magnets	$0.6300(17)^*$		0.325(2)
PFT	6,7-l MZM[3]	0.6304(13)	0.034(3)	0.326(1)
	$O(\epsilon^5)$ exp[3]	0.6290(25)	0.036(5)	0.326(3)
Lattice	HT exp[1]	0.63012(16)	0.0364(2)	0.3265(1)
	MC[2]	0.63020(12)	0.0368(2)	0.3267(1)

* By using the hyperscaling relation $\alpha = 2 - 3\nu$. [1] M. Campostrini, A. Pelissetto, P. Rossi, EV (2002). [2] Y. Deng, HWJ Blöte, (2003) [3] R. Guida, J. Zinn-Justin, (1998)

3D XY universality class

$\mathcal{L} = |\partial_\mu \varphi|^2 + r\,|\varphi|^2 + u\,|\varphi|^4$ (complex φ)

The superfluid transition in ^{4}He is an exceptional experimental opportunity, exploiting also a microgravity environment using the Space Shuttle (data up to a few nK from T_c)



		α	ν	η
EXPT	^{4}He [1]	$-0.0127(3)$	$0.6709(1)^*$	
PFT	6,7-l MZM[2]	$-0.011(4)$	$0.6703(15)$	$0.035(3)$
	$O(\epsilon^5)$ exp[2]	$-0.004(11)$	$0.6680(35)$	$0.038(5)$
Lattice	MC+HT [3]	$-0.0151(3)^*$	$0.6717(1)$	$0.0381(2)$
	MC [4]	$-0.0151(9)^*$	$0.6717(3)$	

* By $\alpha = 2 - 3\nu$. [1] J.A. Lipa, etal, PRB 68 (2003) 174518; PRL 76 (1996) 944. [2] R. Guida, J. Zinn-Justin, (1998). [3] M. Campostrini, M. Hasenbusch, A. Pelissetto, EV (2006). [4] E. Burovski, etal, (2006)

→ Significant discrepancy between **EXPT** and **Lattice** results

There are also several critical phenomena which are described by more general multi-parameter Φ^4 theories:

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{N} (\partial_\mu \varphi_i)^2 + r_i \varphi_i^2] + \frac{1}{4!} \sum_{ijkl=1}^{N} u_{ijkl}\, \varphi_i \varphi_j \varphi_k \varphi_l$$

- The parameter r_i and u_{ijkl} depend on the symmetry.
- If criticality is driven by one T-like parameter, and all φ_i become critical, $\sum_i \varphi_i^2$ must be the only invariant quadratic term. Thus $r_i = r$, $\sum_i u_{iikl} \propto \delta_{kl}$, etc...
- In the absence of a large symmetry like $O(N)$, several quartic couplings must be considered.
- all Φ^4 theories are expected to be trivial for $D = 4$ like $\mathrm{O}(N)$ models

Physically interesting LGW Φ^4 theories

- O(M)⊗O(N) model, fields ϕ_a are M sets of N-comp vectors

$$\mathcal{L} = \sum_a [(\partial_\mu \phi_a)^2 + r\phi_a^2] + u_0 (\sum_a \phi_a^2)^2 + v_0 \sum_{a,b} (\phi_a \cdot \phi_b)^2$$

For $M = 2, N = 3$, $v_0 < 0$, $\mathrm{U}(2) \to \mathrm{O}(2)$, superfluid transitions in ^{3}He.

For $M = 2$, $v_0 > 0$, $O(2) \otimes O(N) \to O(2) \otimes O(N-2)$, noncollinear frustrated magnets (stacked triangular antiferromagnets).

- MN model with a real $M \times N$ matrix field ϕ_{ai}

$$\mathcal{L} = \sum_{i,a} \left[(\partial_\mu \phi_{ai})^2 + r\phi_{ai}^2 \right] + \sum_{ij,ab} (u_0 + v_0 \delta_{ij}) \, \phi_{ai}^2 \phi_{bj}^2$$

For $N \to 0$, disordered spin systems at magnetic transitions.

For $M = 1$, $N = 2, 3$, magnets with cubic anisotropy.

- Spin-density wave model (Φ_a are complex N-comp vectors)

$$|\partial_\mu \Phi_1|^2 + |\partial_\mu \Phi_2|^2 + r(|\Phi_1|^2 + |\Phi_2|^2) + u_{1,0}(|\Phi_1|^4 + |\Phi_2|^4)$$
$$+u_{2,0}(|\Phi_1^2|^2 + |\Phi_2^2|^2) + w_{1,0}|\Phi_1|^2|\Phi_2|^2 + w_{2,0}|\Phi_1 \cdot \Phi_2|^2 + w_{3,0}|\Phi_1^* \cdot \Phi_2|^2$$

Critical behavior in spin-density wave systems.

Quantum transitions in 2D in high-T_c superconductors (cuprates).

- $\mathrm{U}(N)\otimes\mathrm{U}(N)$ models (Φ is a complex N×N matrix)

$$\mathcal{L}_U = \mathrm{Tr}\partial_\mu \Phi^\dagger \partial_\mu \Phi + r\mathrm{Tr}\Phi^\dagger\Phi + u_0\left(\mathrm{Tr}\Phi^\dagger\Phi\right)^2 + v_0 \mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2$$

Finite-T transition in QCD with N quarks, neglecting $\mathrm{U}(1)_A$ anomaly

- $\mathrm{SU}(N)\otimes\mathrm{SU}(N)$ models: $\mathcal{L}_{SU} = \mathcal{L}_U + w_0\left(\det\Phi^\dagger + \det\Phi\right)$

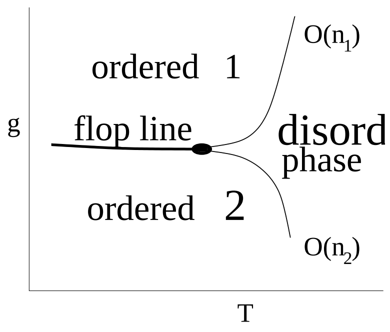
Finite-T transition in QCD taking into account the $\mathrm{U}(1)_A$ anomaly effects

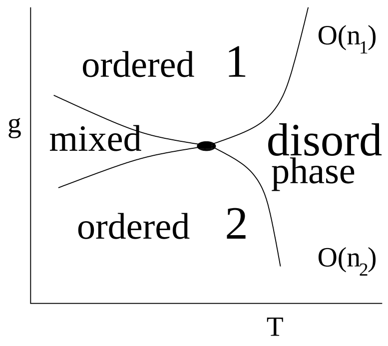
The **competition of different orderings** is described by more complicated LGW Φ^4 theories containing more quadratic invariants

- $O(n_1)\oplus O(n_2)$ theory with two $O(n_1)$ and $O(n_2)$ vector fields

$$\mathcal{L} = (\partial_\mu \vec{\phi}_1)^2 + (\partial_\mu \vec{\phi}_2)^2 + r_1 \vec{\phi}_1^2 + r_2 \vec{\phi}_2^2 + u_1 (\vec{\phi}_1^2)^2 + u_2 (\vec{\phi}_2^2)^2 + w \vec{\phi}_1^2 \vec{\phi}_2^2$$

Multicritical behavior arising from the competition of orderings with symmetries $O(n_1)$ and $O(n_2)$, at the point where the transition lines meet. This point is approached by tuning 2 relevant parameters.

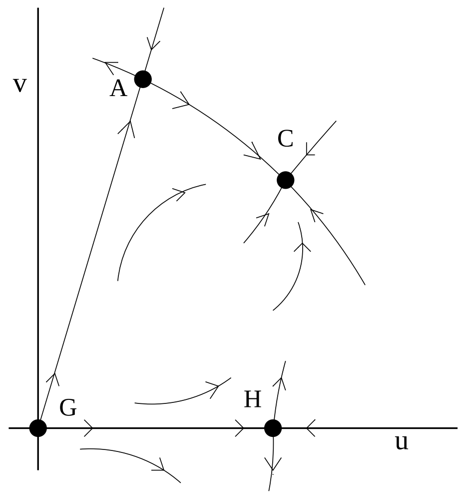




In high-T_c superconductors, anisotropic antiferromagnets, etc...

• The RG flow is determined by the FPs, i.e. common zeroes g^*_{ijkl} of $\beta_{ijkl}(g_{abcd}) \equiv m\partial g_{ijkl}/\partial m$ (MZM), $\beta_{ijkl}(g_{abcd}) \equiv \mu\partial g_{ijkl}/\partial\mu$ ($\overline{\rm MS}$)

A FP is stable if all eigenvalues of its stability matrix $S_{ij} = \partial\beta_i/\partial g_j|_{g=g^*}$ have positive real part

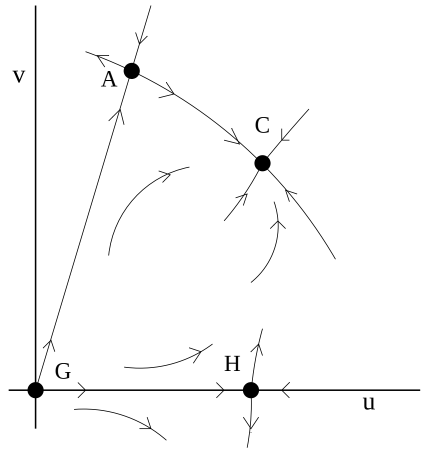


• The existence of a **stable FP** implies that • physical systems with the given global properties can undergo a continuous transition, • the asymptotic behavior in continuous transitions is controlled by the stable FP (apart from cases requiring further tunings)

• The absence of a stable FP predicts 1st-order transitions between the disordered and ordered phases in all systems

• Even in the presence of a stable FP, systems that are outside its attraction domain undergo 1st-order transitions

Multiparameter Φ^4 theories have usually several FPs. An interesting question concerns the existence of a physical quantity such that the comparison of its values at the FPs identifies the most stable FP



In 2D unitary QFT, the **central charge** c is such a quantity

$$\langle T(z_1)T(z_2)\rangle = \frac{c}{2z_{12}^4}, \quad f(L)|_{L\times\infty} = f_\infty L - c\frac{\pi}{6L} \quad \text{at } \; T = T_c$$

***c*-theorem** (Zamolodchikov, 1986): the stable FP in a 2D unitary QFT is the one with the least value of c

No extension to higher dimensions has been achieved yet.

There have been some proposals especially for 4D QFT, but no conclusive results

η for Φ^4 theories $\rightarrow$ η **conjecture** (EV, Zinn-Justin, 2006): *In general unitary Φ^4 theories the infrared stable FP is the one that corresponds to the fastest decay of correlations*

This is the FP with the largest value of the critical exponent η which characterizes the power-law decay of the two-point correlation function $G(x)$ at criticality

$$G(x) \propto \frac{1}{x^{d-2+\eta}}$$

$$d_\Phi = (d-2+\eta)/2$$

- It holds in the case of the $O(N)$-symmetric Φ^4 theory. For $d<4$, the Gaussian FP, for which $\eta=0$, is unstable against the non-trivial Wilson–Fisher FP for which $\eta>0$
- It is proven within the $\varepsilon \equiv 4-d$ expansion, i.e. close to $d=4$
- It remains a conjecture in $d<4$, its validity is confirmed by analytic and numerical results (no counterexample exists!)

RG flow, critical exponents, etc..., by **FT perturbative methods**

$$\mathcal{L} = \sum_i [(\partial_\mu \varphi_i)^2 + r_i \varphi_i^2] + \sum_{ijkl} u_{ijkl}\, \varphi_i \varphi_j \varphi_k \varphi_l$$

• **Massive (disordered-phase) MZM scheme**: expansion in powers of the MZM quartic couplings g_{ijkl}

$$\Gamma^{(2)}_{ij}(p) = \delta_{ij} Z_\varphi^{-1} \left[m^2 + p^2 + O(p^4) \right], \qquad \Gamma^{(4)}_{ijkl}(0) = m\, Z_\varphi^{-2}\, g_{ijkl}$$

• **Massless (critical) $\overline{\rm MS}$ scheme:** Minimal subtraction within the dimensional regularization, ϵ expansion, $d = 3$ $\overline{\rm MS}$ exp

• High-order computations for several LGW Φ^4 theories, to six loops, requiring the calculation $\gtrsim 1000$ diagrams (Pelissetto, EV)

• Resummation exploiting Borel summability and calculation of the large-order behavior, by instanton semiclassical calculation

• The comparison of MZM and $\overline{\rm MS}$ expansions checks the results

O(M)⊗O(N) theory (ϕ_a are M sets of N-component vectors)

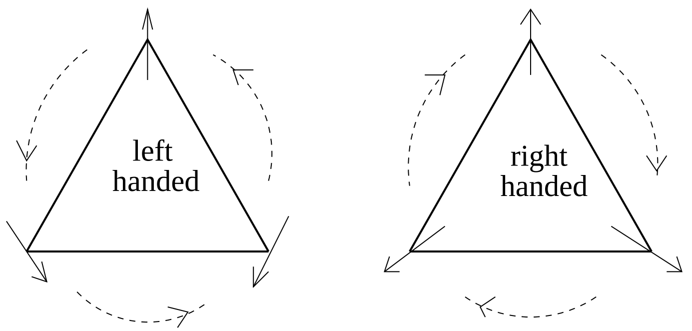
$$\mathcal{L} = \sum_a [(\partial_\mu \phi_a)^2 + r\phi_a^2] + u_0 (\sum_a \phi_a^2)^2 + v_0 \sum_{a,b} [(\phi_a \cdot \phi_b)^2 - \phi_a^2 \phi_b^2]$$

- For $M = 2$, $v_0 > 0$, $O(N) \to O(N-2)$

→ transitions in noncollinear frustrated spin systems, stacked triangular antiferromagnets, such as $CsMnBr_3$, $CsVBr_3$, modeled by

$$H = J_\| \sum_{\langle vw \rangle_{xy}} \vec{s}(v) \cdot \vec{s}(w) - J_\perp, \sum_{\langle vw \rangle_z} \vec{s}(v) \cdot \vec{s}(w) + D \sum_v s_z(v)^2$$

Frustration arise from the special geometry → ordered ground state with a chiral 120^o structure



- For $M = 2, N = 3$, $v_0 < 0$, $\mathrm{U}(2) \to \mathrm{O}(2)$, transition in ^{3}He

Experiments on several physical systems show continuous transitions, with exponents $\nu = 0.57(3), 0.54(3)$ for $N = 2$ and $\nu = 0.62(5)$ for $N = 3$

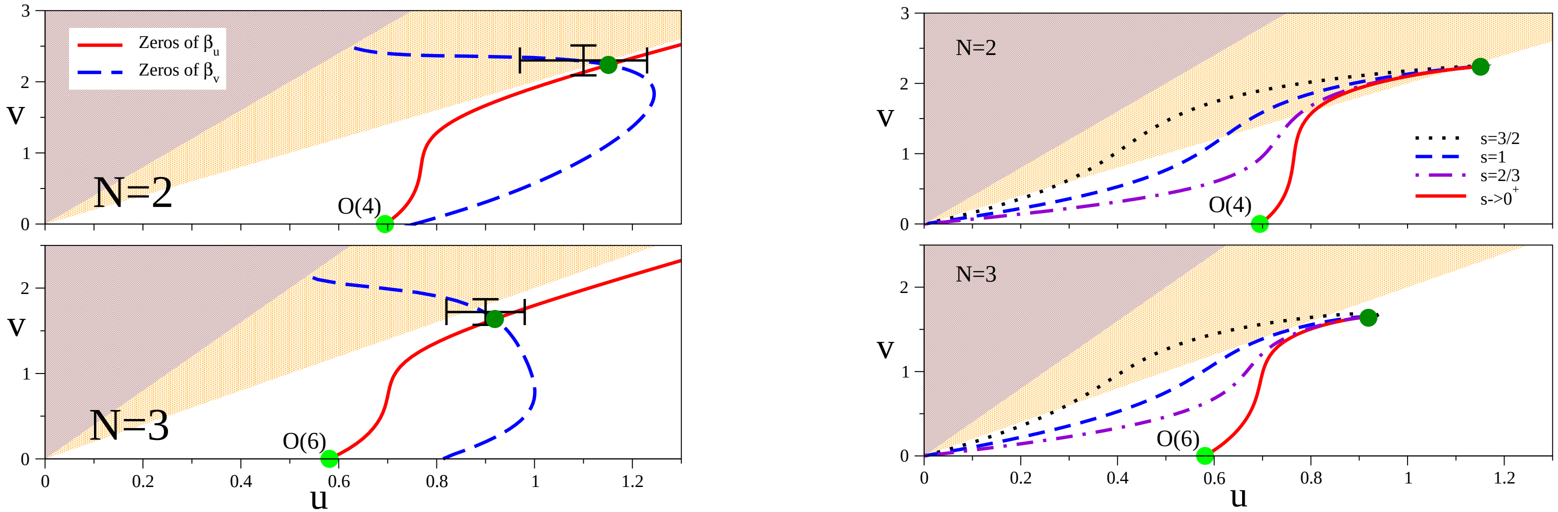
The existence of these new chiral universality classes requires a stable FP in the 3-d RG flow of the O(2)⊗O(N) Φ^4 theory.

Originally investigated by $\epsilon \equiv 4 - d$ expansion: no stable FP is found close to $d = 4$, thus predicting first-order transitions for all systems.

→ **apparent contradiction**, **but** the extension to $d = 3$ is not guaranteed: new FP's may appear going from $d \lesssim 4$ to $d = 3$.

This contradiction has been solved by high-order computations within 3D FT schemes (Pelissetto, Rossi, EV, 2001; Calabrese etal 2004), 6-loop and 5-loop in the MZM and $\overline{\text{MS}}$ schemes respectively , which show the existence of a stable FP.

Their analyses show a stable FP, supporting the existence of new 3-D chiral universality classes, which explain experiments. Critical exponents $\nu = 0.57(3)$ for $N = 2$ and $\nu = 0.55(3)$ for $N = 3$ are in agreement with experiments.



Zeroes of the $\overline{\mathrm{MS}}$ β functions (left) and RG trajectories in the u, v plane from the Gaussian to the stable chiral FP (right).

Magnetic transitions in disordered spin systems

Spin models with impurities: mixing of antiferromagnetic materials with non magnetic ones, $Fe_u Zn_{1-u} F_2$, $Mn_u Zn_{1-u} F_2$ (uniaxial), $Fe_x Er_z$, $Fe_x Mn_y Zr_z$ (isotropic), ^{4}He in porous materials. **Modeled by** $\mathcal{H} = -J \sum_{\langle ij \rangle} \rho_i \, \rho_j \; \vec{s}_i \cdot \vec{s}_j$, where $\rho_i = 1, 0$ with probability p and $1 - p$ respectively.

Quenched disorder: the relaxation of impurities is very slow, thus the free energy $F(\rho) \propto \ln Z(\rho)$ must be averaged over the disorder, thus thermal and then disorder averages

$$\langle \mathcal{O} \rangle(\beta, \{\rho\}) = \frac{\sum_{\{s\}} \mathcal{O} e^{-\beta \mathcal{H}(s;\rho)}}{\sum_{\{s\}} e^{-\beta \mathcal{H}(s;\rho)}}, \qquad \overline{\langle \mathcal{O} \rangle} = \int [d\rho] P(\rho) \langle \mathcal{O} \rangle(\beta, \{\rho\})$$

General **magnetic transitions in the case disorder does not break $O(N)$ symmetry**, such as the $\pm J$ Edwards-Anderson model $\mathcal{H} = -\sum_{\langle ij \rangle} J_{ij} \vec{s}_i \cdot \vec{s}_j$ with $P(J_{ij}) = p\delta(J_{ij} - J) + (1 - p)\delta(J_{ij} + J)$

In the FT approach they can be described by a Φ^4 theory with quenched disorder coupled to the energy-density operator

$$\mathcal{H}_\psi = \partial_\mu \vec{\varphi}(x)^{\,2} + (r + \psi(x))\vec{\varphi}(x)^{\,2} + g_0(\vec{\varphi}(x)^{\,2})^2$$

$\psi(x)$ is a spatially uncorrelated random field, $P(\psi) \sim \exp(-\psi^2/4w)$

The replica trick, $\ln Z = \lim_{n\to 0}(Z^n - 1)/n$, allows us to obtain a translation invariant Hamiltonian (MN model)

$$\mathcal{H}_{MN} = \sum_{i,a}\left[(\partial_\mu \phi_{a,i})^2 + r\phi_{a,i}^2\right] + \sum_{ij,ab}\left(u_0 + v_0\delta_{ij}\right)\phi_{a,i}^2\phi_{b,j}^2$$

$a, b = 1, ...M,\ i, j = 1, ...N,\ u_0 < 0.$

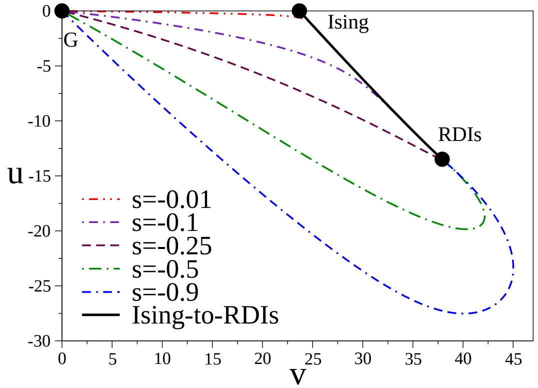
The original system is recovered in the limit $N \to 0$

The critical behavior is determined by the RG flow of the MN model in the limit $N \to 0$, and in particular by analyzing the high-order MZM and $\overline{\rm MS}$ series for $N = 0$.

The critical behavior of the pure system is stable if $\alpha_{\rm pure} < 0$ (Harris, 1974), as in multicomponent systems.

In Ising-like systems the pure Ising FP is unstable since $\alpha_{\rm Is} = 0.1096(5)$. Another stable FP exists, implying the existence of a new 3D RDI universality class.

Experiments confirm it.



^{4}He in porous materials and isotropic magnets show the same critical behavior as pure systems. Ising-like systems behave differently, showing $\nu > \nu_{\rm Ising} \simeq 0.630$

RDI exp	ν	β
experiments [1]	0.69(1)	0.359(9)
PFT[2]	0.678(10)	0.349(5)
MC RSIM [3]	0.683(2)	0.354(1)
MC $\pm J$ IM [3]	0.682(3)	0.353(2)

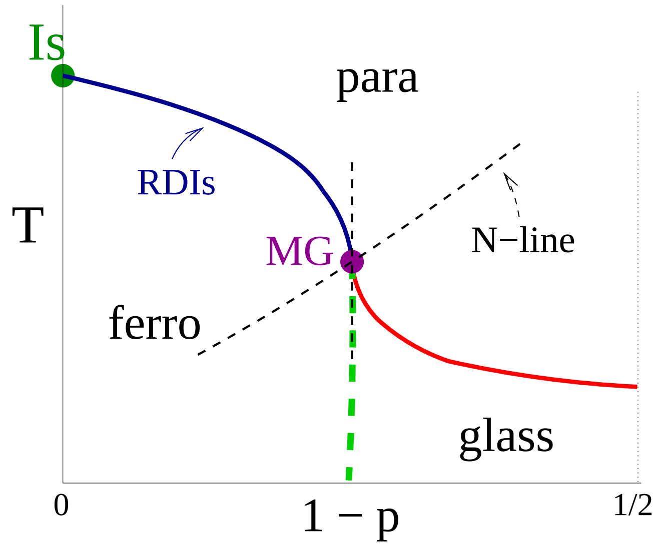
[1] Uniaxial antiferromagnets $Fe_xZn_{1-x}F_2$ (Slanič etal, 1999) [2] To six loops (Pelissetto, EV, 2000) [3] Finite-size scaling analysis of MC data (Hasenbusch, Parisen Toldin, Pelissetto, EV 2007)

The para-ferromagn. transition in **the 3D $\pm J$ Edwards-Anderson Ising model** belong to the RDI universality class.

$H = -\sum_{\langle xy \rangle} J_{xy}\sigma_x\sigma_y$, on a simple cubic lattice, where $\sigma_x = \pm 1$, and $J_{xy} = \pm 1$ are uncorrelated quenched random variables with probability distribution $P(J_{xy}) = p\delta(J_{xy} - 1) + (1-p)\delta(J_{xy} + 1)$.

Simplified model for disordered uniaxial materials which show glassy behavior in their phase diagram, such as $Fe_xMn_{1-x}TiO_3$.

The high-T phase is **paramagnetic**. The low-T phase depends on p: it is **ferromagnetic** for small values of $1-p$, while it is **glassy with vanishing magnetization** for larger values. **High-T** and **low-T** phase are separated by **para-ferro** and **para-glassy** transition lines, which meet at a **magnetic-glassy MCP**



Finite-T transition of QCD with N_f light quarks

$$\mathcal{L}_{\rm QCD} = -\frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{f=1}^{N_f} \bar{\psi}_f \left(i\gamma_\mu D_\mu - m_f \right) \psi_f$$

$$Z = {\rm Tr}\, e^{-\beta H} = \int DAD\bar{\psi}D\psi \; \exp(-S/g^2), \qquad S = \int_0^\beta dt \int d^3x \, \mathcal{L}_{\rm QCD}$$

Chiral symmetry for $m_f = 0$: $\psi_{L,R} \to U(N_f)_{L,R}\psi_{L,R}$

$U(N_f)_L \otimes U(N_f)_R \;\; \simeq U(1)_V \otimes U(1)_A \otimes SU(N_f)_L \otimes SU(N_f)_R$

$U(1)_V$ quark-number conservation, $U(1)_A$ broken by the anomaly, $SU(N_f)_L \otimes SU(N_f)_R$ broken to $SU(N_f)_V$ due to a nonzero $\langle \bar{\psi}\psi \rangle$

Phase transition at $T_c \simeq 200$ Mev restoring chiral symmetry

- Order parameter $\to$ $\Phi_{ij} = \bar{\psi}_{L,i}\psi_{R,j}$, a $N_f \times N_f$ complex matrix
- SB due to $\langle \bar{\psi}\psi \rangle$: $SU(N_f)_L \otimes SU(N_f)_R \to SU(N_f)_V$

The nature of the transition depends on N_f

One may also consider **aQCD**, $SU(N_c)$ gauge theories with N_f Dirac fermions in the adjoint repr (asymptotically free for $N_f < 11/4$)

$$\mathcal{L}_{\rm aQCD} = \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_{f=1}^{N_f} \bar{\psi}_f \gamma_\mu D_\mu \psi_f , \qquad D^{ab}_\mu = \partial_\mu \delta^{ab} + i A^c_\mu f^{ab}_c$$

• Chiral symmetry: $\mathrm{U}(2N_f)$, actually $\mathrm{SU}(2N_f)$ due to the anomaly. Bilinear quark condensation reduces it to $\mathrm{SO}(2N_f)$.

• $\mathrm{Z}(N_c)$, broken in the high-T phase like pure gauge theory

Two phase transitions in aQCD: first-order deconfinement transition at T_d related to $\mathrm{Z}(N_c)$ and then chiral transition at T_c related to $\mathrm{SU}(2N_f)$, $T_c/T_d \approx 8$ for $N_f = 2$

SB at the chiral transition: $\mathrm{SU}(2N_f) \to \mathrm{SO}(2N_f)$ with a $2N_f \times 2N_f$ symmetric complex matrix as order parameter.

The nature of the finite-T transition in QCD can be investigated using renormalization-group methods based on universality (originally applied by Pisarski, Wilczek, 1984)

(A) Let us assume that the transition is continuous, when $\xi \gg 1/T_c$ the system is effectively 3D, then its critical behavior should belong to a 3D universality class characterized by the same symmetry breaking pattern

QCD: $\mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R \to \mathrm{SU}(N_f)_V$, with complex $N_f \times N_f$ matrix order parameter

aQCD: $\mathrm{SU}(2N_f) \to \mathrm{SO}(2N_f)$, with complex symmetric $N_f \times N_f$ matrix order parameter

If $\mathrm{U}(1)_A$ is effectively restored at T_c, then

QCD: $\mathrm{U}(N_f)_L \otimes \mathrm{U}(N_f)_R \to \mathrm{U}(N_f)_V$ [aQCD: $\mathrm{U}(2N_f) \to \mathrm{O}(2N_f)$]

(B) The most general 3D LGW Φ^4 theory compatible with the given symmetry breaking pattern provides an effective theory of the critical modes at T_c. Neglecting $\mathrm{U}(1)_A$ anomaly,

$$\mathcal{L}_{\mathrm{U}(N)} = \mathrm{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r\mathrm{Tr}\Phi^\dagger\Phi + u_0 \left(\mathrm{Tr}\Phi^\dagger\Phi\right)^2 + v_0 \mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2$$

- if Φ_{ij} is a complex $N \times N$ matrix, $v_0 > 0$,
$\mathrm{U}(N)_L \otimes \mathrm{U}(N)_R \rightarrow \mathrm{U}(N)_V$, corresponding to QCD
- if Φ_{ij} is symmetric, $\mathrm{U}(N) \rightarrow \mathrm{O}(N)$, corresponding to aQCD$_{N/2}$
- Due to anomaly: $\mathrm{SU}(N_f)_L \otimes \mathrm{SU}(N_f)_R \rightarrow \mathrm{SU}(N_f)_V$ for QCD [$\mathrm{SU}(2N_f) \rightarrow \mathrm{O}(2N_f)$ for aQCD], achieved by

$$\mathcal{L}_{\mathrm{SU}(N)} = \mathcal{L}_{\mathrm{U}(N)} + w_0 \left(\mathrm{det}\Phi^\dagger + \mathrm{det}\Phi\right)$$

- Nonvanishing quark masses correspond to an external field H_{ij} coupled to Φ_{ij}, by adding $+\mathrm{Tr}\left(H\Phi + \mathrm{h.c.}\right)$

(C) A necessary condition of consistency with the initial hypothesis **(A)** of continuous transition is the existence of a stable FP in the corresponding 3D Φ^4 theory

• If no stable FP's exist, the transition of QCD (aQCD) is predicted to be first order

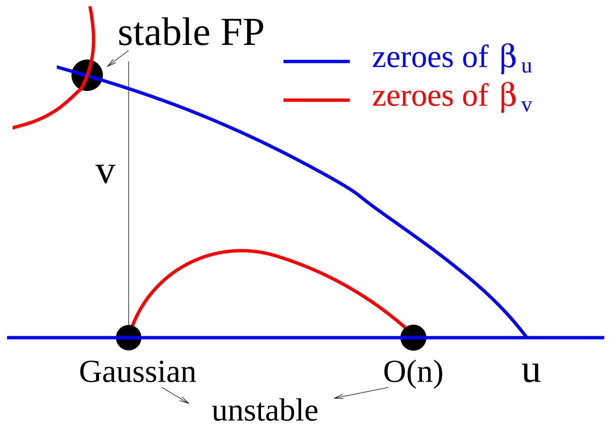
• If a stable 3D FP is found, the transition can be continuous, in this case its universal critical behavior is determined by the FP. But, it may still be first order if the system is outside the attraction domain of the stable FP.

• $N = 2$, $\mathrm{QCD}_{N_f=2}$, no anomaly: The corresponding universality class exists if there is a stable FP in the 3D U(2)⊗U(2) theory with a complex 2×2 matrix field Φ

$$\mathcal{L}_{\mathrm{U}(N)} = \mathrm{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r\mathrm{Tr}\Phi^\dagger\Phi + u_0 \left(\mathrm{Tr}\Phi^\dagger\Phi\right)^2 + v_0 \mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2$$

• No stable FP is found close to 4D within the ϵ expansion (Pisarski, Wilczek, 1984), its extension to 3D would predict always 1st-order

• In both MZM and 3D $\overline{\mathrm{MS}}$ schemes, the analysis of high-order series show the presence of a stable FP (Basile, Pelissetto, EV 2005)



Thus, the 3-D U(2)⊗U(2)/U(2) universality class exists, which implies that transition can be continuous, with $\nu \approx 0.7$, $\eta \approx 0.1$

• $\mathrm{QCD}_{N_f=2}$ taking into account the $\mathrm{U}(1)_A$ anomaly

Symmetry breaking $\rightarrow$ $\mathrm{SU}(2)\otimes\mathrm{SU}(2)/\mathrm{SU}(2) \simeq \mathrm{O}(4)/\mathrm{O}(3)$

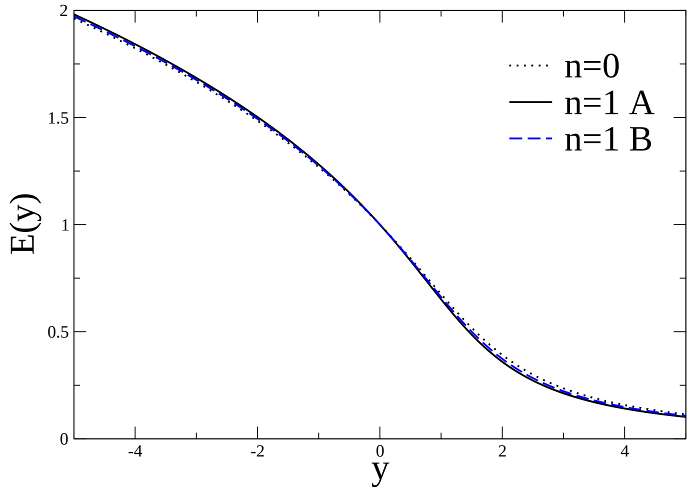
This corresponds to the O(4) universality class. This means that, if the transition is continuous, it must show the O(4) scaling behavior

$$\vec{M} \propto \vec{H}|H|^{(1-\delta)/\delta}E(y)$$

$$y \propto t|H|^{-1/(\beta+\delta)}$$

$$\langle\bar{\psi}\psi\rangle \propto |M|, \quad m_f \propto |H|$$

$$\delta = 4.789(6), \quad \beta = 0.3882(10)$$

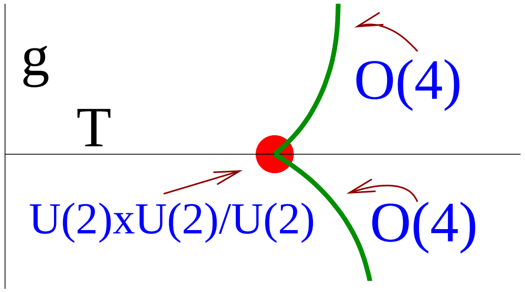


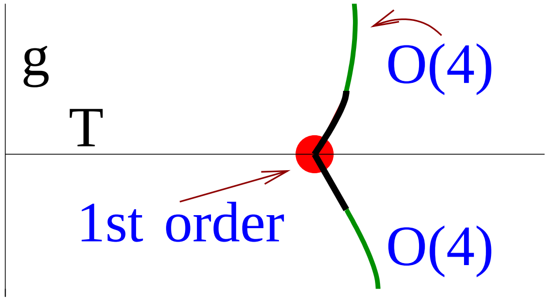
$\rightarrow$ The corresponding LGW Φ^4 theory is more complicated $\rightarrow$

Due to $U(1)_A$ anomaly: $\mathcal{L}_{\rm SU(2)} = \mathcal{L}_{\rm U(2)} + w_0 \left(\det\Phi^\dagger + \det\Phi\right) + x_0 \left({\rm Tr}\Phi^\dagger\Phi\right)\left(\det\Phi^\dagger + \det\Phi\right) + y_0 \left[(\det\Phi^\dagger)^2 + (\det\Phi)^2\right]$, where w_0,x_0,$y_0$$\sim g$ $\rightarrow$ effective breaking of $\mathrm{U}(1)_A$ (Pelissetto, EV, 2005)

2 mass terms: transition lines in the T-g plane meeting at a MCP controlled by the $\mathrm{U}(2)_L\otimes\mathrm{U}(2)_R$ theory for $g = 0$

in the case of continuous (left) or first order (right) at $g = 0$





O(4) critical behavior if the transition is continuous and $g \neq 0$. A mean-field behavior may occur for particular values of g in the right case.

If $|g|$ is small (a suppression of anomaly effects around T_c is suggested by MC), we may have crossover effects controlled by the $\mathrm{U}(2)\otimes\mathrm{U}(2)$ MCP [$\mathcal{F}_{\rm sing} \approx t^{3\nu} f(g t^{-\phi})$, $\nu \approx 0.7$, $\phi \approx 1.5$]

• QCD with $N_f \geq 3$

$$\mathcal{L} = \mathrm{Tr}(\partial_\mu \Phi^\dagger)(\partial_\mu \Phi) + r\mathrm{Tr}\Phi^\dagger\Phi + u_0 \left(\mathrm{Tr}\Phi^\dagger\Phi\right)^2 + v_0 \mathrm{Tr}\left(\Phi^\dagger\Phi\right)^2 + w_0 \left(\det\Phi^\dagger + \det\Phi\right)$$

The high-order analysis does not show any stable FP for $N \geq 3$, therefore the transition in QCD with $N_f \geq 3$ is predicted to be first order

• aQCD with $N_f = 2$

A stable FP is found when Φ is a 4×4 symmetric matrix field, corresponding to SU(4)/SO(4), showing the existence of a corresponding 3D universality class. (Basile, Pelissetto, EV, 2005)

Summary of predictions for the finite-T transition of QCD

QCD

no anomaly, $N_c \to \infty$

	$SU(N_f) \otimes SU(N_f)$	$U(N_f) \otimes U(N_f)$
$N_f = 1$	crossover or first order	O(2) or first order
$N_f = 2$	O(4) or first order	$U(2)_L \otimes U(2)_R / U(2)_V$ or first order
$N_f \geq 3$	first order	first order

adjoint QCD

no anomaly

	$SU(2N_f)/SO(2N_f)$	$U(2N_f)/O(2N_f)$
$N_f = 1$	O(3) or first order	U(2)/O(2) or first order
$N_f = 2$	SU(4)/SO(4) or first order	first order

Comparison with lattice MC results

HARD TASK: the chiral transition must be studied in the continuum and massless limit, i.e. $V \to \infty$, $N_t \to \infty$, $m_f \to 0$.

Physical case: $N_f = 2$ **light quarks + heavier quarks**

MC simulations show a crossover at $m_f > 0$ around their physical values (Bernard etal, Cheng et al, Aoki et al, de Forcrand et al,)

The nature of the transition in the chiral limit is still of some interest: some scaling relations of the critical behavior may still be valid at the physical values of the quark masses.

• QCD $N_f = 2$: from RG $\rightarrow$ O(4) or first order

STILL CONTROVERSIAL: Some MC results favor a continuous transition. But the results have not been sufficient to settle its O(4) nature yet. There are also results favoring a first-order transition.

(many refs should be cited ...)

Two remarks: $\rightarrow$ A continuous transition implies a crossover for $m_f > 0$, as apparently shown by MC, while a first-order transition is expected to persist at $m_f > 0$, $\rightarrow$ It depends on the mass of the quark s: for sufficiently small mass of s it is expected to be first order as in the case of $N_f = 3$.

• QCD $N_f \geq 3$ no stable FP, thus $\rightarrow$ 1^{st} order, confirmed by MC

(e.g., de Forcrand, Philipsen, Karsch, etc...,)

• adjoint-QCD $N_f = 2$ $\rightarrow$ SU(4)/SO(4) or 1^{st} order. MC results are consistent with a continuous transition (Karsch etal, Engels etal) some conclusions ...

• **The RG flows of generalized Φ^4 theories is physically interesting because they describe many critical phenomena.**

• **Field-theory approaches are effective, even in complex cases with several quadratic and quartic parameters**

• **Accurate results are obtained by perturbative expansions and high-order computations, after resummation exploiting Borel summability and the knowledge of their large-order behavior. Satisfactory comparisons with experiments**

Results from PFT may improve by extending the series. This is a very hard numerical task, essentially limited by the computation of the multivariable integrals associated with the huge number of diagrams (they are already $\gtrsim 1000$ at six loops).