# Boundary Field Theory approach to the Renormalization of SQUID devices

D. Giuliano[*,†], P. Sodano[%,+]

[*] INFN - Gruppo Collegato di Cosenza
[†] Dipartimento di Fisica, Università della Calabria
[%] INFN - Sezione di Perugia
[%] Dipartimento di Fisica, Università di Perugia

# Content of the talk

- **Circuits made with Josephson junctions connected by means of thin superconducting wires: the rf-SQUID:**
  1. **The weakly coupled regime: sinusoidal Josephson current;**
  2. **The strongly-coupled regime: impossibility of disentangle the junctions from the "environmental modes" of the wires.**
- **1+1 boundary Field theory description of the rf-SQUID: physical constraints and boundary conditions:**
  1. **Primary fields and corresponding boundary interactions for the rf-SQUID;**
  2. **Scaling analysis near the weakly coupled fixed point;**
  3. **Scaling analysis near the strongly coupled fixed point.**
- **A simple multi-junction generalization: the dc-SQUID:**
  1. **Primary fields and corresponding boundary interactions for the dc-SQUID;**
  2. **Scaling analysis near the weakly coupled fixed point;**
  3. **Scaling analysis near the strongly coupled fixed point.**
- **Conclusions and further perspectives:**
  1. **Josephson devices $\rightarrow$ 1+1-dimensional boundary field theories for quantum wires with impurities/boundaries;**

2. **Engineering multi-wire/multi-impurity devices ⇒ finite coupling IR-stable fixed points in the phase diagram ⇒ multi junction devices with finite coupling, IR-stable fixed points;**
3. **Perspectives: possible use as flux qubits.**

# Circuits made with Josephson junctions connected by means of thin superconducting wires.

- **rf-SQUID**

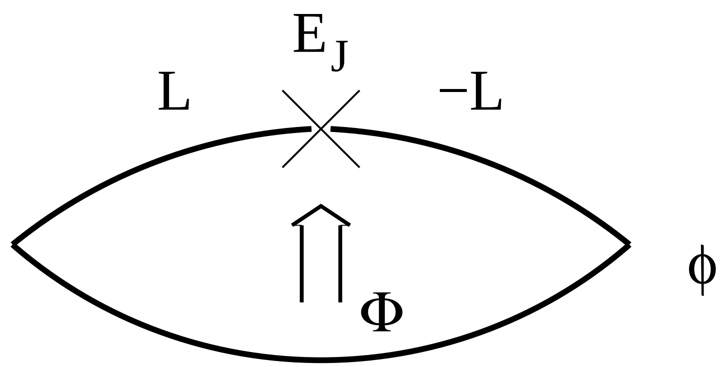


Figure 1: The rf-SQUID.

1. **Lagrangian:**

$$L = \int_{-L}^{L} dx \left\{ \frac{\hbar^2}{2e_c} \left( \frac{\partial \Phi}{\partial t} \right)^2 - \frac{\hbar^2 n_s S}{4m} \left( \frac{\partial \Phi}{\partial x} - \frac{\Phi}{2L} \right)^2 \right\}$$

$$+ E_J \cos[\Phi(L,t) - \Phi(-L,t)]$$

2. $n_s$ **is the superfluid density,** $m$ **is the electron mass,** $1/e_c$ **is the characteristic inverse charging energy per unit length of the loop,** $S = r \times r$ **is the cross section of the wire,** $\epsilon$ **is the dielectric constant of the medium the loop is embedded within,** $R$ **is the distance from a metallic screen,** $E_J$ **is the Josephson energy of the junction.**

3. **Luttinger parameters and spinless Luttinger Hamiltonian:**

**Charging energy of the lead + inductive energy of the lead + Josephson energy of the junction** $\Rightarrow$

$$H_{\rm rf} = \frac{g}{4\pi}\int_{-L}^{L} dx \left[\frac{1}{u}\left(\frac{\partial\Phi(x,t)}{\partial t}\right)^2 + u\left(\frac{\partial\Phi(x,t)}{\partial x}\right)^2\right]$$

$$-E_J \cos[\Phi(L,t) - \Phi(-L,t) + \varphi]$$

**with** $g = \pi\sqrt{n_s S/(m e_c)}$**,** $u = \sqrt{n_s S e_c/(4m)}$**,** $\varphi = \Phi/\Phi_0^*$**,** $\Phi_0^* = 2e/h$**,** $\Phi(x,t) \to \Phi(x,t) - \varphi x/(2L)$**.**

- **dc-SQUID**

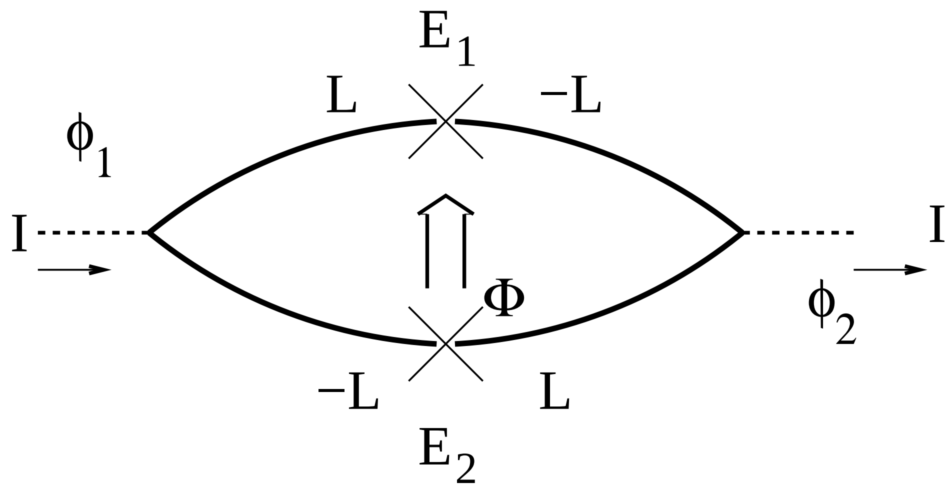


Figure 2: The dc-SQUID.

- **The weakly-coupled regime**

1. **rf-SQUID:** $u(\Phi_0^*)^2/L/E_J \gg 1 \Rightarrow$ **large inductive energy of the ring** $\Rightarrow$ **"freezing" the collective excitations (plasmons) in the phase field** $\Phi(x,t)$

$$\Phi(L,t) - \Phi(-L,t) \approx 0 \mod 2\pi \Rightarrow E[\varphi] \approx -E_J \cos(\varphi)$$

- **Strongly coupled regime**

$u(\Phi_0^*)^2/L/E_j \leq 1 \Rightarrow$ **strong effect of phase fluctuations of the order parameter** $\Rightarrow$ **field theory approach.**

# Boundary Conformal Field Theory description of the rf-SQUID

- **Basic fields of the noninteracting theory**

1. **Bosonic field and chiral components**

$$\Phi(x,t) = \frac{1}{\sqrt{2g}}[\phi_R(x-ut) + \phi_L(x+ut)]$$

**Mode expansion for the chiral fields**

$$\phi_{L/R}(x \mp ut) = q_{L/R} \mp \frac{2\pi p_{L/R}}{L}(x \mp ut) + i\sum_{n\neq 0}\frac{\alpha_{L/R}(n)}{n}e^{ik_n(x\mp ut}$$

**with**

$$[q_{L/R}, p_{L/R}] = i \;\; ; \;\; [\alpha_{L/R}(n), \alpha_{L/R}(n')] = \mp n\delta_{n+n',0}$$

**and**

$$p_{L/R}|\mathrm{GS}\rangle = \alpha_R(n)|\mathrm{GS}\rangle = \alpha_L(-n)|\mathrm{GS}\rangle = 0 \;\; (n>0)$$

**Mode expansion for** $\Phi(x,t)$

$$\Phi(x,t) = \phi_0 - \frac{2\pi}{(2L)}Px + \frac{2\pi}{(2L)}\frac{\tilde{P}}{g}ut +$$

$$\frac{i}{\sqrt{2g}}\sum_{n\neq 0}\left[\frac{\alpha_R(n)}{n}e^{ik_n(x-ut)} + \frac{\alpha_L(n)}{n}e^{ik_n(x+ut)}\right] \quad .$$

**Dual field**

$$\Theta(x,t) = \sqrt{\frac{g}{2}}[\phi_R(x-ut) - \phi_L(x+ut)] \quad ,$$

**Mode expansion for $\Theta(x,t)$**

$$\Theta(x,t) = \theta_0 - \frac{2\pi}{(2L)}\tilde{P}x + \frac{2\pi}{(2L)}gPut +$$

$$i\sqrt{\frac{g}{2}}\sum_{n\neq 0}\left[\frac{\alpha_R(n)}{n}e^{ik_n(x-ut)} - \frac{\alpha_L(n)}{n}e^{ik_n(x+ut)}\right] \quad .$$

<u>**Notice**</u>

$$g\frac{\partial\Phi(x,t)}{\partial x} = \frac{1}{u}\frac{\partial\Theta(x,t)}{\partial t} \quad ; \quad \frac{\partial\Phi(x,t)}{\partial x} = \frac{1}{gu}\frac{\partial\Theta(x,t)}{\partial t}$$

**Current density**

$$j(x,t) = \sqrt{2}gu\frac{\partial\Phi(x,t)}{\partial x}$$

**Charge density**

$$\rho(x,t) = \sqrt{2}\frac{\partial\Theta(x,t)}{\partial x} = \frac{\sqrt{2}g}{u}\frac{\partial\Phi(x,t)}{\partial t}$$

**Hamiltonian**

$$H = \frac{\pi u}{L}\left[\frac{P^2}{g} + g(\tilde{P})^2\right] + \frac{2\pi u}{L}\sum_{n=1}^{\infty}[\alpha_R(-n)\alpha_R(n) + \alpha_L(n)\alpha_L(-n)]$$

**Two-boundary theory ($x = L, -L$).**

2. **Primary fields**

$$V_{Q,\tilde{Q}}(x,t) =: \exp[i\tilde{Q}\Phi(x,t) + iQ\Theta(x,t)] :$$

**Basic commutators**

$$\left[-\frac{\partial\Phi(x,t)}{\partial x}, V_{Q,\tilde{Q}}(x',t)\right] = 2\pi Q\delta(x-x')V_{Q,\tilde{Q}}(x',t)$$

**and**

$$\left[-\frac{\partial\Theta(x,t)}{\partial x}, V_{Q,\tilde{Q}}(x',t)\right] = 2\pi \tilde{Q}\delta(x-x')V_{Q,\tilde{Q}}(x',t)$$

$\Rightarrow V_{Q,\tilde{Q}}(x,t)$ **changes the charge ($\tilde{P}$) by an amount $\propto \tilde{Q}$, the current ($P$) by an amount $\propto Q$.**

3. **Boundary conditions:**

**Energy conservation**

$$\frac{dE_{\mathrm{rf}}}{dt} = 0 \quad ; \quad \left[\frac{1}{u}\frac{\partial^2}{\partial t^2} - u\frac{\partial^2}{\partial x^2}\right]\Phi(x,t) = 0$$

$\Rightarrow$ **boundary conditions for** $\Phi(x,t)$

$$\frac{gu}{2\pi}\frac{\partial\Phi(L,t)}{\partial x} + E_J \sin[\Phi(L,t) - \Phi(-L,t) + \varphi] = 0$$

**and**

$$\frac{gu}{2\pi}\frac{\partial\Phi(-L,t)}{\partial x} + E_J \sin[\Phi(L,t) - \Phi(-L,t) + \varphi] = 0$$

**Charge conservation (continuity of the current)** $\Rightarrow$

$$\frac{\partial\Phi(L,t)}{\partial x} - \frac{\partial\Phi(-L,t)}{\partial x}$$

$\Rightarrow$ **the tunnel boundary interaction can only depend on** $\Phi(L,t) - \Phi(-L,t)$

4. **Construction of boundary operators**

$$x \to \pm L \Rightarrow V_{Q,\tilde{Q}}(x,t) \longrightarrow V^{(B)}(t)$$

**The allowed boundary operators depend on the boundary conditions (M. Oshikawa, C. Chamon, I. Affleck, "Delayed Evolution of Boundary Conditions" - DEBC).**

- ## The weakly coupled fixed point

1. **Neumann-Neumann boundary conditions:**

$$E_J \to 0 \Rightarrow \frac{\partial \Phi(L,t)}{\partial x} = \frac{\partial \Phi(-L,t)}{\partial x} = 0$$

**Consequences**

$$P = 0 \quad ; \quad \begin{array}{l} e^{ik_n L}\alpha_R(n) + e^{-ik_n L}\alpha_L(-n) = 0 \\ e^{-ik_n L}\alpha_R(n) + e^{ik_n L}\alpha_L(-n) = 0 \end{array}$$

**implies**

$$k_n = \frac{\pi n}{2L} \ (n \in \mathbf{Z}) \quad ; \quad \alpha_R(n) = (-1)^{n-1}\alpha_L(-n) \equiv \alpha(n)$$

2. **Boundary fields and boundary operators:**

**Boundary fields**

$$\Phi(L,t) = \phi_0 + \frac{2\pi}{(2L)}\frac{\tilde{P}}{g}ut + i\sqrt{\frac{2}{g}}\sum_{n\neq 0} i^n \frac{\alpha(n)}{n} e^{-i\frac{\pi n}{(2L)}ut}$$

**and**

$$\Phi(-L,t) = \phi_0 + \frac{2\pi}{(2L)}\frac{\tilde{P}}{g}ut + i\sqrt{\frac{2}{g}}\sum_{n\neq 0} (-i)^n \frac{\alpha(n)}{n} e^{-i\frac{\pi n}{(2L)}ut}$$

**(Primary field) Boundary operators**

$$V^{(N)}_{\tilde{Q},\pm}(t) =: \exp[\pm i\tilde{Q}\Phi(\pm L,t)] :$$

**<u>Remark</u>: N-N Boundary conditions $\Rightarrow$**

$$\frac{\partial \Theta(L,t)}{\partial t} = \frac{\partial \Theta(-L,t)}{\partial t} = 0$$

$\Rightarrow$ **No boundary operators depending on $\Theta$.**

3. **Correlation functions and scaling dimensions**

**Correlation functions**

$$\langle V^{(N)}_{\tilde{Q},\pm}(t) V^{(N)}_{\tilde{Q}',\pm}(t') \rangle \sim_{u(t-t')/L \ll 1} \frac{\delta_{q+q',0}}{\left[\frac{u}{L}(t-t')\right]^{\frac{2Q^2}{g}}}$$

$$\langle V^{(N)}_{\tilde{Q},\pm}(t) V^{(N)}_{\tilde{Q}',\mp}(t') \rangle \sim_{u(t-t')/L \ll 1} 1$$

**Scaling dimensions**

$$h^{(N)}_{\tilde{Q},\pm} = \frac{(\tilde{Q})^2}{g}$$

4. **Josephson energy operator (junction energy - interaction representation)**

$$H_B = -E_J : \cos[\Phi(L,t) - \Phi(-L,t) + \varphi] :=$$

$$-\frac{E_J e^{i\varphi}}{2} W_1(t) - \frac{E_J e^{-i\varphi}}{2} W_{-1}(t)$$

**with**

$$W_n(t) = V^{(N)}_{n,+}(t) V^{(N)}_{-n,-}(t)$$

**Scaling dimension of $W_n(t)$**

$$H_n = 2n^2/g$$

**Additional interactions produced by O.P.E.'s, with higher periodicity in $\varphi$, according to**

$$W_n(t)W_{n'}(t') \approx_{t'\to t} \left[\frac{\pi u}{2L}(t-t')\right]^{-H_n-H_{n'}+H_{n+n'}} W_{n+n'}(t')$$

## 5. Effective boundary interaction and running coupling strengths

$$\tilde{H}_J = -\frac{1}{2}\sum_{n=1}^{\infty}\left\{E_n e^{in\varphi}W_n(t) + E_n e^{-in\varphi}W_{-n}(t)\right\}$$

**Running couplings**

$$g_n = \left(\frac{L}{2\pi a}\right)^{1-\frac{2n^2}{g}} E_n$$

**with $a \sim \xi$ short-distance cutoff.**

**Renormalization group equations**

$$\frac{dg_1}{d\ln(L/L_0)} = \beta_1(g_1,g_2) = \left(1-\frac{2}{g}\right)g_1 + g_1 g_2$$

$$\frac{dg_2}{d\ln(a/a_0)} = \beta_2(g_1,g_2) = \left(1-\frac{8}{g}\right)g_2 + (g_1)^2$$

**Solutions**

$$g_1(L) \approx g_1(L_0)\left(\frac{L}{L_0}\right)^{\left(1-\frac{2}{g}\right)} ; g_2(L) \approx \frac{(g_1^2(L))}{1+4/g}\left[1-\left(\frac{L}{L_0}\right)^{-1-\frac{4}{g}}\right]$$

6. **Stability of the N-N fixed point:**

$g < 2 \Rightarrow$ **irrelevant interaction (stable weakly coupled fixed point): the theory is perturbative in** $E_J$ **and, thus**

$$I[\Phi] = -\frac{1}{c}\left[\frac{\partial\langle H_B\rangle}{\partial\Phi}\right] = \frac{2eE_J}{c}\left(\frac{2\pi a}{L}\right)^{\frac{2}{g}} \sin\left[\frac{\Phi}{\Phi_0^*}\right]$$

**the same as from the "classical" analysis.**
**For** $g > 2$ **all the higher harmonics become relevant (for instance** $(g_2(L)/g_1(L)) \propto \left(\frac{L}{L_0}\right)^{1-\frac{2}{g}}$**.**
**The scaling stops at** $\bar{L}_*$ **at which** $g_1(\bar{L}_*) \sim 1$**, that is**

$$\bar{L}_* = 2\pi a \left(\frac{u}{aE_J}\right)^{\frac{g}{g-1}}$$

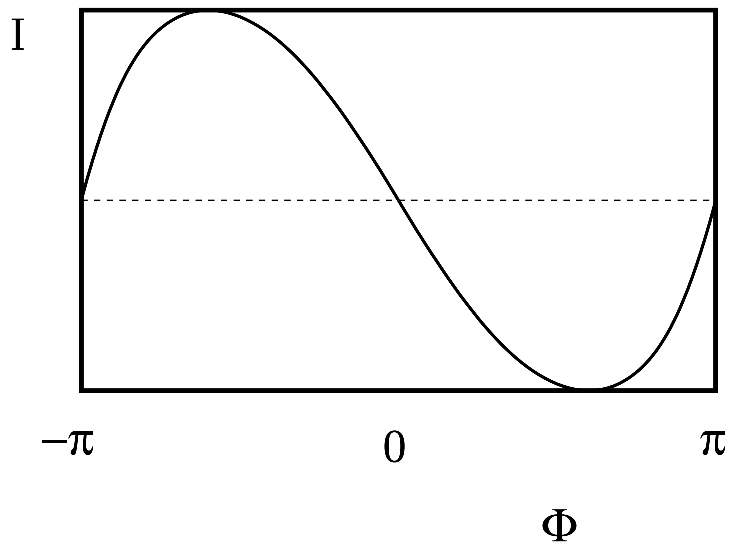


Figure 3: The Josephson current in the perturbative regime around the N-N fixed point.

- **The strongly coupled fixed point**

  1. **Dirichlet boundary conditions:**
     $E_J \to \infty \Rightarrow$

     $$\Phi(L,t) - \Phi(-L,t) + \varphi = 2\pi k \quad ; \quad k \in \mathbf{Z}$$

     **This implies**

     $$\sin(k_n L)[\alpha_R(n) + \alpha_L(-k_n)] = 0$$

     **and the eigenvalues of $P$, $\{P_k\}$**

     $$P_k = -k - \frac{\varphi}{2\pi}$$

     **The boundary condition**

$$\frac{\partial}{\partial x}[\Phi(L,t) - \Phi(-L,t)] = 0$$

**implies**

$$\sin(k_n L)[\alpha_R(n) - \alpha_L(-k_n)] = 0$$

**Thus**

$$k_n = \frac{\pi}{L} n \quad ; \quad n \in \mathbf{Z}$$

2. **Partition function at the strongly coupled fixed point**

$$\mathcal{Z} = \mathrm{Tr}[e^{-\beta H_{\mathrm{rf}}}] = \mathcal{Z}_{\tilde{P}} \mathcal{Z}[\varphi]$$

**The factor** $\mathcal{Z}_{\tilde{P}} = \mathrm{Tr}[e^{-\beta \frac{\pi u}{L} g(\tilde{P})^2}]$ **contains no relevant informations concerning the boundary dynamics**

$$\mathcal{Z}[\varphi] = \frac{1}{\eta^2(q)} \sum_{k \in Z} \exp\left[-\beta \frac{g\pi u}{L}\left(-\frac{\varphi}{2\pi} + k\right)^2\right]$$

**where** $\eta(x) = \prod_{n=1}^{\infty}(1 - x^n)$**, and** $q = \exp\left[-\beta \frac{\pi u}{L}\right]$**.**
**Josephson current**

$$I[\varphi] = -\lim_{\beta \to \infty} \frac{1}{\beta} \frac{\partial \ln \mathbf{Z}_D[\varphi]}{\partial \varphi} \propto \varphi - [\varphi]$$

**where** $[\varphi]$ **is the integer part of** $\varphi$ **(in units of** $2\pi$**) ("sawtooth-like" behavior.**

3. **Relevant perturbation at the strongly coupled fixed point**

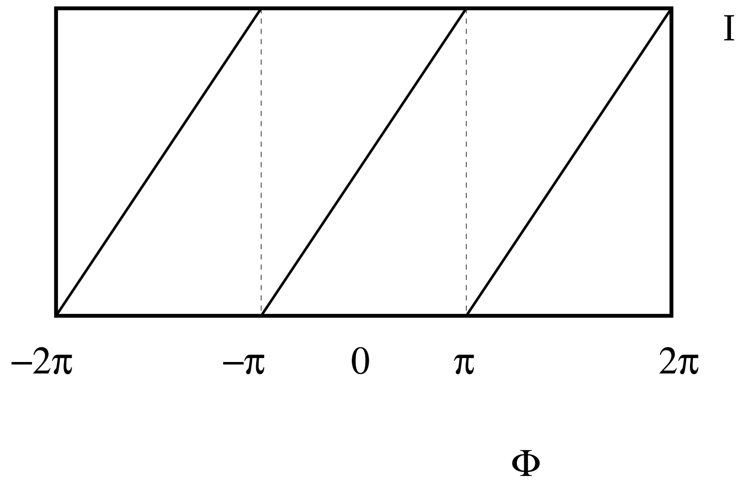


Figure 4: The Josephson current in the perturbative regime at the strongly coupled fixed point.

**Dirichlet boundary conditions**

$$\frac{\partial}{\partial t}[\Phi(L,t) - \Phi(-L,t)] = 0$$

**$\Rightarrow$ no $W_n$-operators.**

**Mode expansion for the dual boundary field**

$$xo\Theta(L,t) = \theta_0 - \pi\tilde{P} + \frac{\pi}{L}Pgut + i\sqrt{g}\sum_{n\neq 0}(-1)^n \frac{\alpha(n)}{n} e^{-i\frac{\pi n}{L}ut}$$

$$\Theta(-L,t) = \Theta(L,t) + 2\pi\tilde{P}$$

**with**

$$\alpha(n) = \frac{1}{\sqrt{2}}[\alpha_R(n) + \alpha_L(-n)]$$

**Setting**

$$\Theta(t) = \frac{1}{2}[\Theta(L,t) + \Theta(-L,t)]$$

$\Rightarrow$ **most relevant boundary operators**

$$V_Q^{(D)}(t) =: \exp[iQ\Theta(t)] :$$

**Most relevant boundary perturbation**

$$\tilde{H}_B = -\lambda[V_1^{(D)}(t) + V_{-1}^{(D)}(t)]$$

4. **Correlation functions, scaling dimensions and renormalization group equations**

**Correlation functions**

$$\langle V_Q^{(D)}(t) V_{Q'}^{(D)}(t') \rangle \sim_{u(t-t')/L \ll 1} \frac{\delta_{\tilde{Q}+\tilde{Q}',0}}{\left[\frac{\pi u(t-t')}{L}\right]^g}$$

**Scaling dimension**

$$h_Q^{(D)} = \frac{gQ^2}{2}$$

**Running couplings**

$$\bar{\lambda} = \left(\frac{L}{2\pi a}\right)^{1-\frac{g}{2}} \lambda$$

**Renormalization group equations**

$$\frac{d\bar{\lambda}}{d\ln(L/L_0)} = \beta_\lambda(\bar{\lambda}) = \left(1 - \frac{g}{2}\right)\bar{\lambda}$$

5. **Physical meaning of $\tilde{H}_B$**

$P = P_k \Rightarrow$ **"zero-mode" (inductive) contribution to the total energy**

$$E_k^{(0)} = \frac{\pi u}{L}\left(k + \frac{\varphi}{2\pi}\right)^2$$

**At** $\varphi = 2\ell\pi + \pi, \ell \in \mathbf{Z}$

$$E_{\ell-1}^{(0)} = E_\ell^{(0)}$$

**For instance,** $\ell = 0 \Rightarrow$ **twofold degenerate energy minimum,** $P_0, P_1$ **(two eigenstates of the persistent Josephson current operator** $P$**)**

**From the basic commutators**

$$\left[P, : e^{iQ\Theta(t)} :\right] = Q : e^{iQ\Theta(t)} :$$

**the operators** $V_1^{(D)}(t), V_{-1}^{(D)}(t)$ **represent "jumps" back and fro the minima, i.e., they represent the (real-time version of) instanton solutions interpolating between the minima.**

6. **Stability of the Dirichlet fixed point**

**Solution of the renormalization group equations**

$$\bar{\lambda}(L) = \bar{\lambda}(L_0)\left(\frac{L}{L_0}\right)^{1-\frac{g}{2}}$$

$g > 2 \Rightarrow \tilde{H}_B$ **is an irrelevant interaction (stable strongly coupled fixed point): the theory is perturbative in** $\lambda$**.**

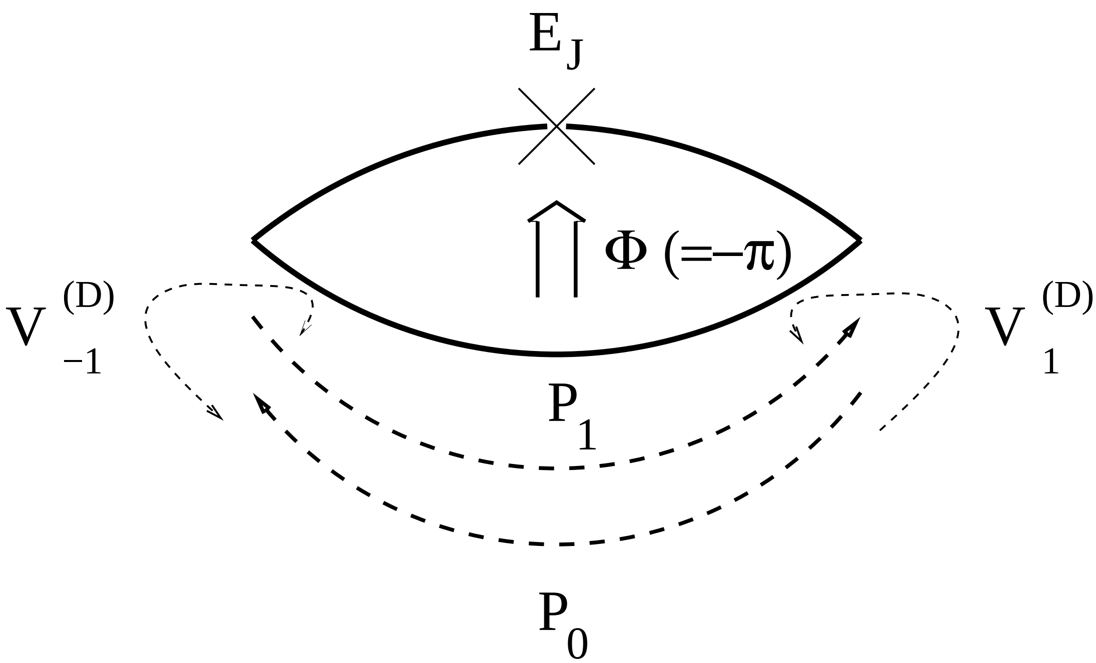


Figure 5: Eigenstates of the current operators for $\ell = 0$.

**For** $g < 2$ $\tilde{H}_B$ **is a relevant interaction (unstable strongly coupled fixed point). This drives the system out of the strongly coupled fixed point.**

7. **Boundary field theory allows for a strightforward description of any phase, no matter how deep might be the entanglement between the junction and the wires**

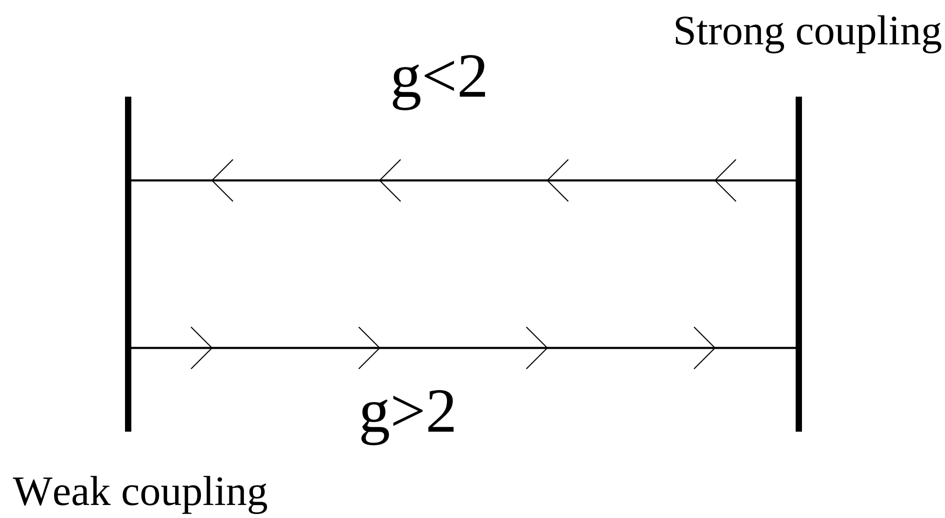


Figure 6: Phase diagram of the rf-SQUID.

# Two-junction generalization

- **Basic fields and boundary interaction**

  1. **"Normal" combinations and Hamiltonian**

$$\Phi_\pm(x,t) = \frac{1}{\sqrt{2}}[\Phi_1(x,t) \pm \Phi_2(-x,t)]$$

**and**

$$\Theta_\pm(x,t) = \frac{1}{\sqrt{2}}[\Theta_1(x,t) \pm \Theta_2(-x,t)]$$

**In terms of these fields**

$$H_{\rm dc} = \frac{g}{4\pi} \sum_{a=\pm} \int_{-L}^{L} dx \left[ \frac{1}{u} \left( \frac{\partial \Phi_a(x,t)}{\partial t} \right)^2 + u \left( \frac{\partial \Phi_a(x,t)}{\partial x} \right)^2 \right]$$

$$-E_1 \cos\left[\sqrt{2}\Phi_-(L) + \frac{\varphi}{2}\right] - E_2 \cos\left[\sqrt{2}\Phi_-(-L) - \frac{\varphi}{2}\right]$$

**Only $\Phi_-$ enters the boundary interaction**

2. **Boundary conditions**

**Current continuity**

$$\frac{\partial}{\partial x}[\Phi_1(\pm L, t) - \Phi_2(\mp L, t)] = 0 \Rightarrow \frac{\partial \Phi_+(\pm L, t)}{\partial x} = 0$$

**Dynamical boundary conditions**

$$\frac{gu}{2\pi}\frac{\partial \Phi_-(L, t)}{\partial x} + \sqrt{2}E_1 \sin\left[\sqrt{2}\Phi_-(L, t) + \frac{\varphi}{2}\right] = 0$$

**and**

$$\frac{gu}{2\pi}\frac{\partial \Phi_-(-L, t)}{\partial x} + \sqrt{2}E_2 \sin\left[\sqrt{2}\Phi_-(-L, t) - \frac{\varphi}{2}\right] = 0$$

- ## The weakly coupled fixed point

1. **Neumann-Neumann boundary conditions**

$$E_1 = E_2 = 0 \Rightarrow \frac{\partial \Phi_-(L,t)}{\partial x} = \frac{\partial \Phi_-(-L,t)}{\partial x} = 0$$

**As for the rf-SQUID at the weakly coupled fixed point, the allowed boundary operators are**

$$V^{(N)}_{\tilde{Q},\pm}(t) =: \exp[i\tilde{Q}\Phi_-(\pm L,t)] :$$

**Since N-N Boundary conditions + current continuity condition $\Rightarrow$**

$$\frac{\partial \Theta_\pm(L,t)}{\partial t} = \frac{\partial \Theta_\pm(-L,t)}{\partial t} = 0$$

**$\Rightarrow$ No boundary operators depending either on $\Theta_+$, or on $\Theta_-$.**

2. **Correlation functions and scaling dimensions**

**Correlation functions**

$$\langle V^{(N)}_{\tilde{Q},\pm}(t) V^{(N)}_{\tilde{Q}',\pm}(t') \rangle \sim_{u(t-t')/L \ll 1} \frac{\delta_{q+q',0}}{\left[\frac{u}{L}(t-t')\right]^{\frac{2Q^2}{g}}}$$

$$\langle V^{(N)}_{\tilde{Q},\pm}(t) V^{(N)}_{\tilde{Q}',\mp}(t') \rangle \sim_{u(t-t')/L \ll 1} 1$$

**Scaling dimensions**

$$h^{(N)}_{\tilde{Q},\pm} = \frac{(\tilde{Q})^2}{g}$$

3. **Boundary interaction operator**

$$H_B = -\frac{E_1 e^{i\frac{\varphi}{2}}}{2} V_{1,+}^{(N)}(t) - \frac{E_1 e^{-i\frac{\varphi}{2}}}{2} V_{-1,+}^{(N)}(t)$$

$$-\frac{E_2 e^{i\frac{\varphi}{2}}}{2} V_{1,-}^{(N)}(t) - \frac{E_1 e^{-i\frac{\varphi}{2}}}{2} V_{-1,-}^{(N)}(t)$$

**Even in this case additional interaction operators (higher harmonics in $\varphi$) are generated by O.P-E.'s as, for instance**

$$W_{n,n'}(t) \approx \lim_{t' \to t} \frac{V_{n,+}^{(N)}(t) V_{n',-}^{(N)}(t')}{(t-t')^{h_n + h_{n'} - h_{n+n'}}}$$

4. **Running couplings and renormalization group equations**

**Running couplings**

$$g_1 = \left( \frac{L}{2\pi a} \right)^{1-\frac{2}{g}} E_1 \quad ; \quad g_2 = \left( \frac{L}{2\pi a} \right)^{1-\frac{2}{g}} E_2$$

**Renormalization group equations**

$$\frac{dg_1}{d\ln(L/L_0)} = \beta_1(g_1) = \left(1 - \frac{2}{g}\right) g_1$$

$$\frac{dg_2}{d\ln(L/L_0)} = \beta_2(g_2) = \left(1 - \frac{2}{g}\right) g_2$$

**Solutions**

$$g_j(L) = g_j(L_0) \left( \frac{L}{L_0} \right)^{1-\frac{2}{g}}$$

5. **Stability of the weakly coupled fixed point**

$g < 2 \Rightarrow$ **both interactions are irrelevant** $\Rightarrow$ **stable weakly coupled fixed point.**
**The theory is perturbative in** $E_j$ **and, thus**

$$I[\Phi] = -\frac{1}{c}\left[\frac{\partial\langle(H_B)^2\rangle}{\partial\Phi}\right] = \frac{2eE_1E_2}{c}\left(\frac{2\pi a}{L}\right)^{\frac{2}{g}}\sin\left[\frac{\Phi}{\Phi_0^*}\right]$$

**the same as for the rf-SQUID, but now** $I \propto E_1E_2$**.**

**For** $g > 2$ **all the higher harmonics become relevant, as for the rf-SQUID.**

**The scaling stops at** $\bar{L}_*$ **determined by the lower one between** $E_1$ **and** $E_2$**, that is**

$$\bar{L}_* = 2\pi a\left(\frac{u}{aE}\right)^{\frac{g}{g-1}}$$

**where** $E = \min\{E_1, E_2\}$**.**

## • The strongly coupled fixed point

1. **Dirichlet-Dirichlet boundary conditions**
   $E_1, E_2 \to \infty \Rightarrow$

$$\sqrt{2}\Phi_-(L,t) + \frac{\varphi}{2} = 2\pi n_1 \quad ; \quad \sqrt{2}\Phi_-(-L,t) + \frac{\varphi}{2} = 2\pi n_2$$

   **Supplementing these boundary conditions with current continuity**

$$\frac{\partial \Phi_+(L,t)}{\partial x} = \frac{\partial \Phi_+(-L,t)}{\partial x} = 0$$

   **allows for constructing the pertinent boundary operators as for the rf-SQUID.**

2. **$\varphi$-depending term in the partition function**

$$\mathcal{Z} = \mathrm{Tr}[e^{-\beta H_{\mathrm{dc}}}] = \mathcal{Z}_{\tilde{P}} \mathcal{Z}[\varphi]$$

   **Again, the factor $\mathcal{Z}_{\tilde{P}} = \mathrm{Tr}[e^{-\beta \frac{\pi u}{2L} g (\tilde{P}_+)^2}]$ contains no relevant informations on the boundary dynamics**

$$\mathcal{Z}[\varphi] = \frac{1}{\eta^2(q)} \sum_{k \in Z} \exp\left[ -\beta \frac{g\pi u}{2L} \left( -\frac{\varphi}{2\pi} + k \right)^2 \right]$$

   **Josephson current**

$$I[\varphi] = -\lim_{\beta \to \infty} \frac{1}{\beta} \frac{\partial \ln \mathbf{Z}_D[\varphi]}{\partial \varphi} \propto \varphi - [\varphi]$$

   **as for the rf-SQUID.**

3. **Most relevant perturbation at the D-D fixed point**

   **The D-D boundary conditions allow for the boundary operators**

$$V^{(D)}_{Q,\pm}(t) =: \exp[i\Theta_+(\pm L, t)] :$$

**Acting with** $V^{(D)}_{Q,+}(t) \Rightarrow n_1 \to n_1 - Q$**,** $n_2$ **unchanged;**
**Acting with** $V^{(D)}_{Q,-}(t) \Rightarrow n_2 \to n_2 - Q$**,** $n_1$ **unchanged.**

4. **Correlators and scaling dimensions**
   **Correlation functions**

$$\langle V^{(D)}_{Q,\pm}(t) V^{(D)}_{Q',\pm}(t') \rangle \sim_{u(t-t')/L \ll 1} \frac{\delta_{\tilde{Q}+\tilde{Q}',0}}{\left[\frac{\pi u(t-t')}{L}\right]^g}$$

$$\langle V^{(D)}_{Q,\pm}(t) V^{(D)}_{Q',\mp}(t') \rangle \sim_{u(t-t')/L \ll 1} \delta_{\tilde{Q}+\tilde{Q}',0}$$

**Scaling dimension**

$$h^{(D)}_Q = \frac{gQ^2}{2}$$

**This time: two different instanton excitations** $(V^{(D)}_{\pm 1,\pm}(t))$ **that make the D-D fixed point unstable for** $g < 2 \Rightarrow$ **same phase diagram as for the rf-SQUID.**

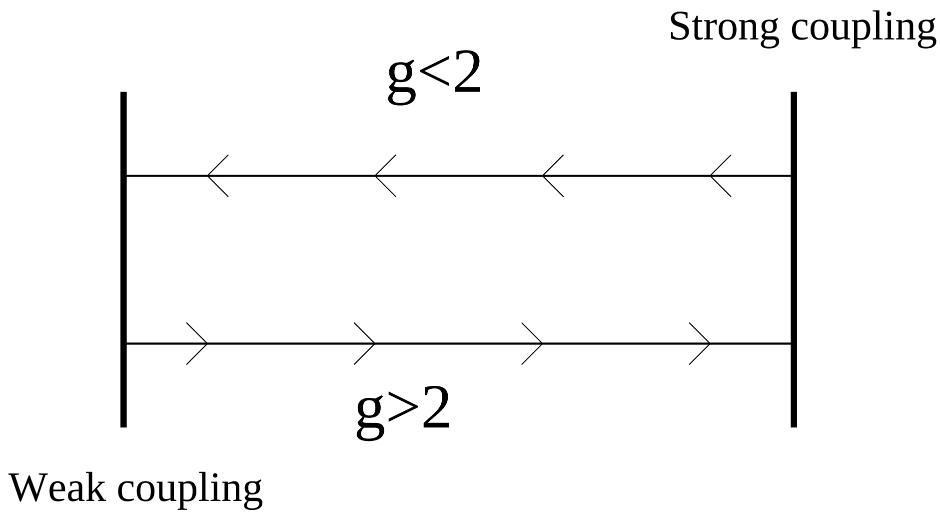


Figure 7: Phase diagram of the dc-SQUID.

# Conclusions and further perspectives

- **1+1-dimensional boundary field theories as a theoretical framework for Josephson devices**

  1. **Superconducting thin wires $\longrightarrow$ Luttinger liquid (1+1 -dimensional CFT)**
  2. **Junction $\longrightarrow$ tunneling between two Luttinger liquids (boundary interaction).**

- **More complex boundary interaction ⇒ finite-coupling IR fixed points (3-wire junction: Affleck, Chamon, Oshikawa)**

  **Fermionic version**

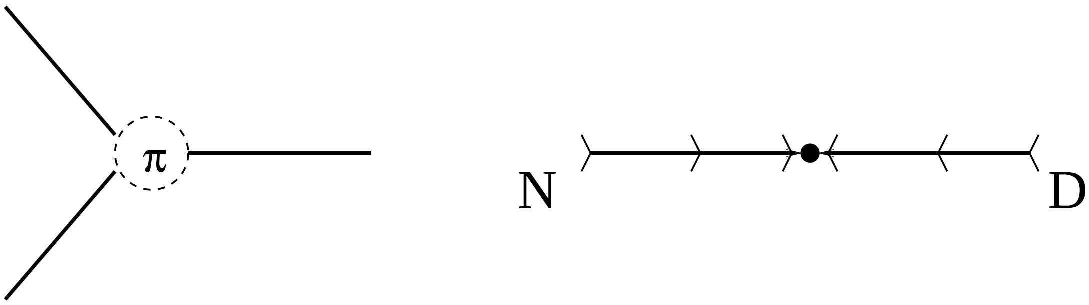


Figure 8: AOC device and phase diagram for $\varphi = \pi$, $1 < g < 3$.

1. **Josephson device ⇒ Bosonic realization of a similar phase diagram**

- **Application: stable qubits with optimal quantum interference**

  1. **"N-D" phase diagram ⇒ either stable D fixed point (2-state qubit), but irrelevant instantons (fragile quantum intereference), or relevant instantons (robust quantum interference), but relevance of higher energy eigenstate from the full spectrum**
  2. **Possible solution: stable fixed point**