# Quasi-stationary states in systems with long-range interactions

STEFANO RUFFO

*Dipartimento di Energetica “S. Stecco” and INFN*

*Center for the Study of Complex Dynamics (CSDC), Firenze*

Statistical Mechanics and Field Theory, Bari

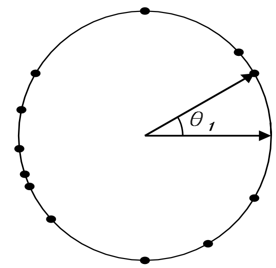
20-22 September 2006

# Plan

- Hamiltonian Mean Field (HMF) model
- Quasi-stationary states
- Non equilibrium phase transition
- Lynden-Bell entropy
- Maximum entropy principle
- Application to the free electron laser

# HMF model

$$H_{XY} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} (1 - \cos(\theta_i - \theta_j))$$



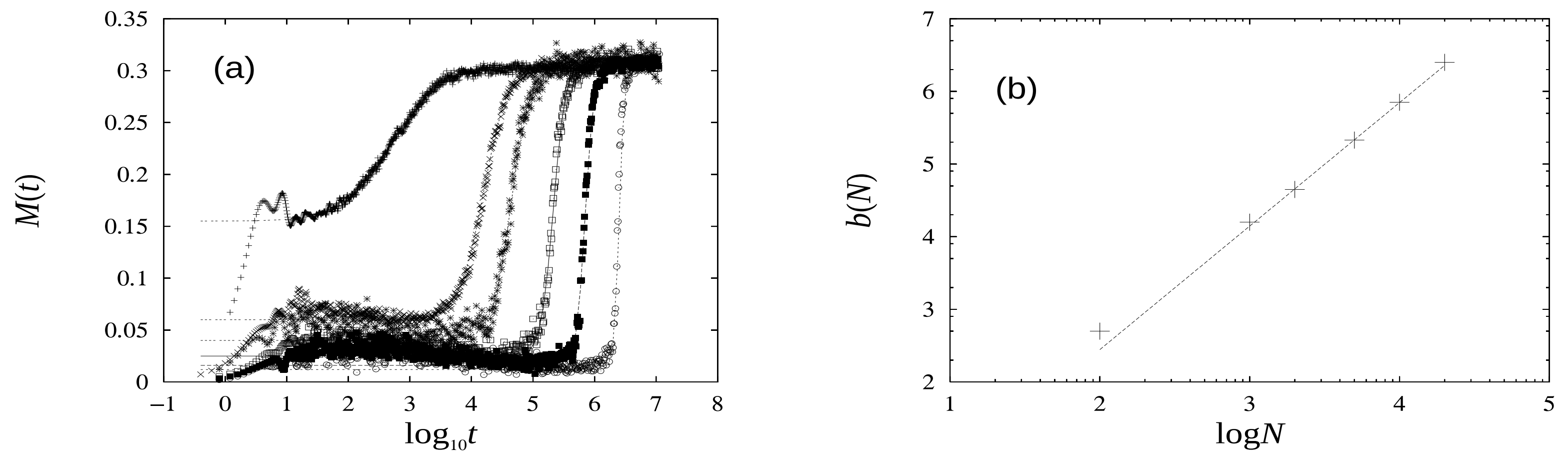

Inspired by

- Gravitational and charged sheet models
- Wave-particle interactions

# Quasi-stationary states

Initial water-bag of (semi) width $\Delta\theta_0 = \pi$ and height $\Delta p_0$ (which determines the energy $U = H/N$)

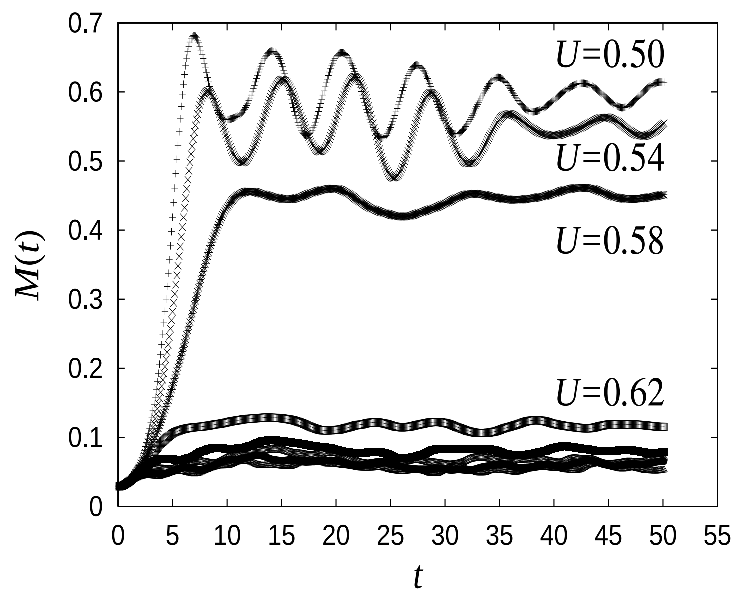
Magnetization $\mathbf{M} = (\sum_{i=1}^{N} \cos\theta_i/N, \sum_{i=1}^{N} \sin\theta_i/N)$

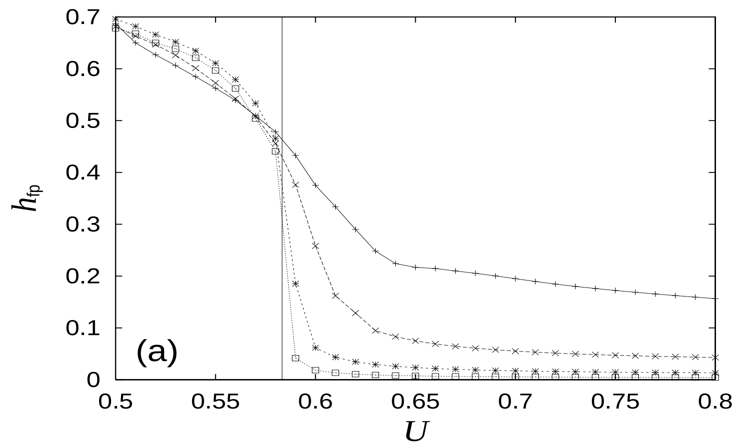



LEFT: $U = 0.69$, from left to right
$N = 10^2, 10^3, 2\times10^3, 5\times10^3, 10^4, 2\times10^4$

RIGHT: Power law increase of the lifetime, exponent $1.7$

# Phase transition




LEFT: $N = 10^3$

RIGHT: First peak height as a function of $U = H/N$ for increasing values of $N$ ($10^2, 10^3, 10^4, 10^5$)

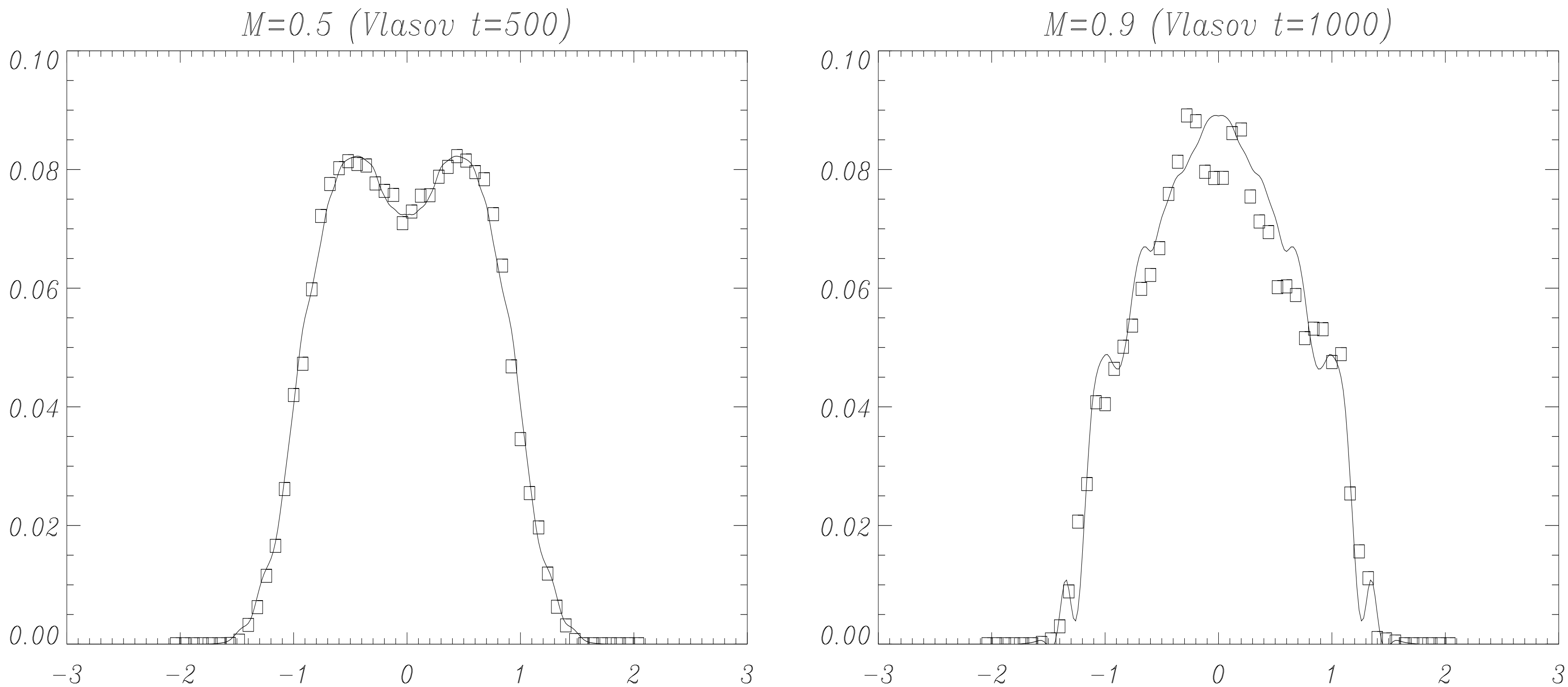
# HMF Vlasov equation

$$\frac{\partial f}{\partial t} + p\frac{\partial f}{\partial \theta} - \frac{dV}{d\theta}\frac{\partial f}{\partial p} = 0 \quad ,$$

$$\begin{aligned} V(\theta)[f] &= 1 - M_x[f]\cos(\theta) - M_y[f]\sin(\theta) \ , \\ M_x[f] &= \int f(\theta,p,t)\ \cos\theta \mathrm{d}\theta\mathrm{d}p \quad , \\ M_y[f] &= \int f(\theta,p,t)\ \sin\theta \mathrm{d}\theta\mathrm{d}p \quad . \end{aligned}$$

Specific energy $e[f] = \int (p^2/2) f(\theta,p,t)\mathrm{d}\theta\mathrm{d}p + 1/2 - (M_x^2 + M_y^2)/2$ and momentum $P[f] = \int p f(\theta,p,t)\mathrm{d}\theta\mathrm{d}p$ are conserved.

# Vlasov simulations



# Lynden-Bell entropy

Assume that the initial distribution $f(\theta, p, 0)$ takes only two values $(0, f_0)$ ("water bag"). Time evolution can only modify the shape of the boundary of the "water-bag", conserving the area inside it. Hence, the distribution remains two-level as time evolves. Coarse-graining amounts to perform a local average of $f$ inside a given cell, which gives $\bar{f}$. The "mixing" entropy per particle associated with $\bar{f}$ is (Lynden-Bell, 1967)

$$s(\bar{f}) = -\int \mathrm{d}p\mathrm{d}\theta \left[ \frac{\bar{f}}{f_0} \ln \frac{\bar{f}}{f_0} + \left(1 - \frac{\bar{f}}{f_0}\right) \ln \left(1 - \frac{\bar{f}}{f_0}\right) \right].$$

Lynden-Bell guesses that the initial evolution ("violent relaxation") is characterized by a maximization of this "fermionic" entropy with given constraints (e.g. energy, momentum, ...).

# Derivation of Lynden-Bell entropy

Fermionic principle: If the initial single-particle distribution is two-level $(0, f_0)$, it remains two level during time evolution (Liouville theorem).

Other invariants (Casimirs) of the Vlasov equation exist.

Divide the bounded phase space into a finite number of macrocells. Each macrocell contains $\nu$ microcells of size $h$. Let $n_i$ be the number of microcells occupied by the level $f_0$ in the $i^{th}$ macrocell and $\mathcal{N}$ the total number of occupied microcells.
Then the number of microstates compatible with the macrostate $n_i$ is

$$W(\{\bar{f}\}) = W(\{n_i\}) = \frac{\mathcal{N}!}{\prod_i n_i!} \times \prod_i \frac{\nu!}{(\nu - n_i)!},$$

with $\bar{f}$ the coarse-grained distribution. The mixing (Lynden-Bell) entropy is

$$S_{LB} = \ln[W(\{\bar{f}\})].$$

# Maximal Lynden-Bell entropy states

$$\bar{f}(\theta, p) = f_0 \frac{e^{-\beta(p^2/2 - M_y[\bar{f}]\sin\theta - M_x[\bar{f}]\cos\theta) - \lambda p - \mu}}{1 + e^{-\beta(p^2/2 - M_y[\bar{f}]\sin\theta - M_x[\bar{f}]\cos\theta) - \lambda p - \mu}}.$$

$$f_0 \quad \frac{x}{\sqrt{\beta}} \int \mathrm{d}\theta e^{\beta \mathbf{M}\cdot\mathbf{m}} F_0\left(x e^{\beta \mathbf{M}\cdot\mathbf{m}}\right) = 1$$

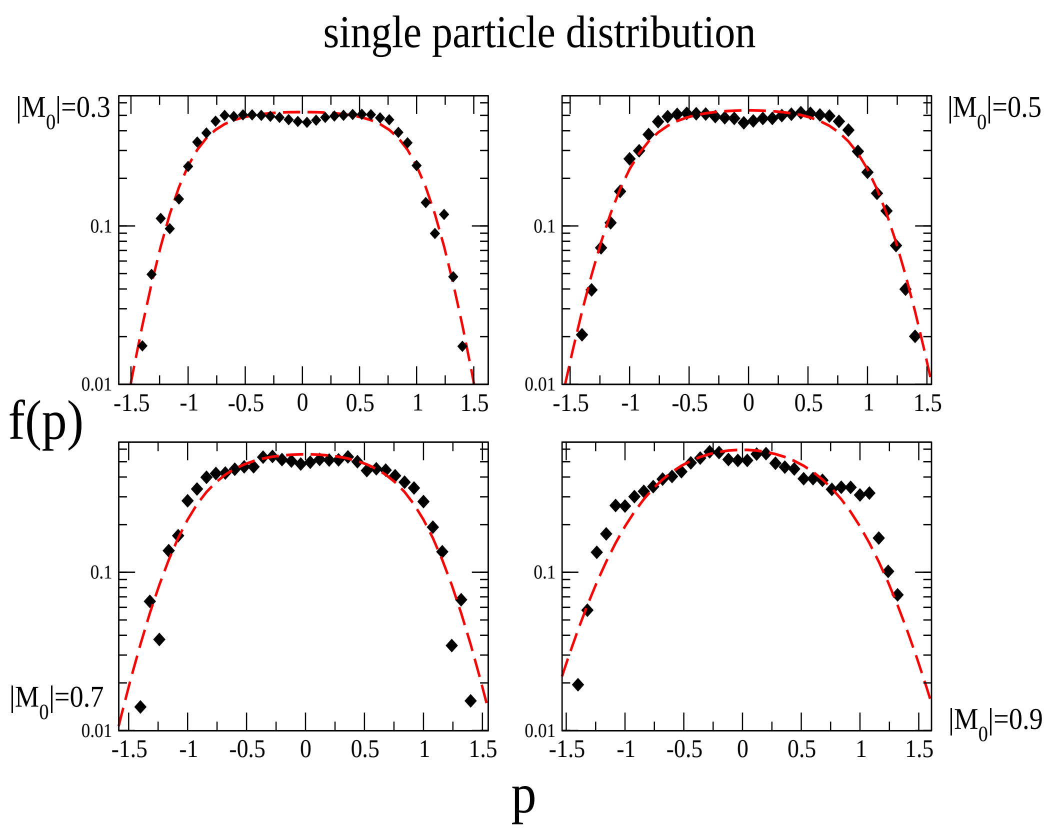
$$f_0 \quad \frac{x}{2\beta^{3/2}} \int \mathrm{d}\theta e^{\beta \mathbf{M}\cdot\mathbf{m}} F_2\left(x e^{\beta \mathbf{M}\cdot\mathbf{m}}\right) = e + \frac{M^2 - 1}{2}$$

$$f_0 \quad \frac{x}{\sqrt{\beta}} \int \mathrm{d}\theta \cos\theta e^{\beta \mathbf{M}\cdot\mathbf{m}} F_0\left(x e^{\beta \mathbf{M}\cdot\mathbf{m}}\right) = M_x$$

$$f_0 \quad \frac{x}{\sqrt{\beta}} \int \mathrm{d}\theta \sin\theta e^{\beta \mathbf{M}\cdot\mathbf{m}} F_0\left(x e^{\beta \mathbf{M}\cdot\mathbf{m}}\right) = M_y$$

$\mathbf{M} = (M_x, M_y)$, $\mathbf{m} = (\cos\theta, \sin\theta)$.
$F_0(y) = \int \exp(-v^2/2)/(1 + y\exp(-v^2/2))\mathrm{d}v$,
$F_2(y) = \int v^2 \exp(-v^2/2)/(1 + y\exp(-v^2/2))\mathrm{d}v$.
$f_0 = 1/(4\Delta\theta_0\Delta p_0)$

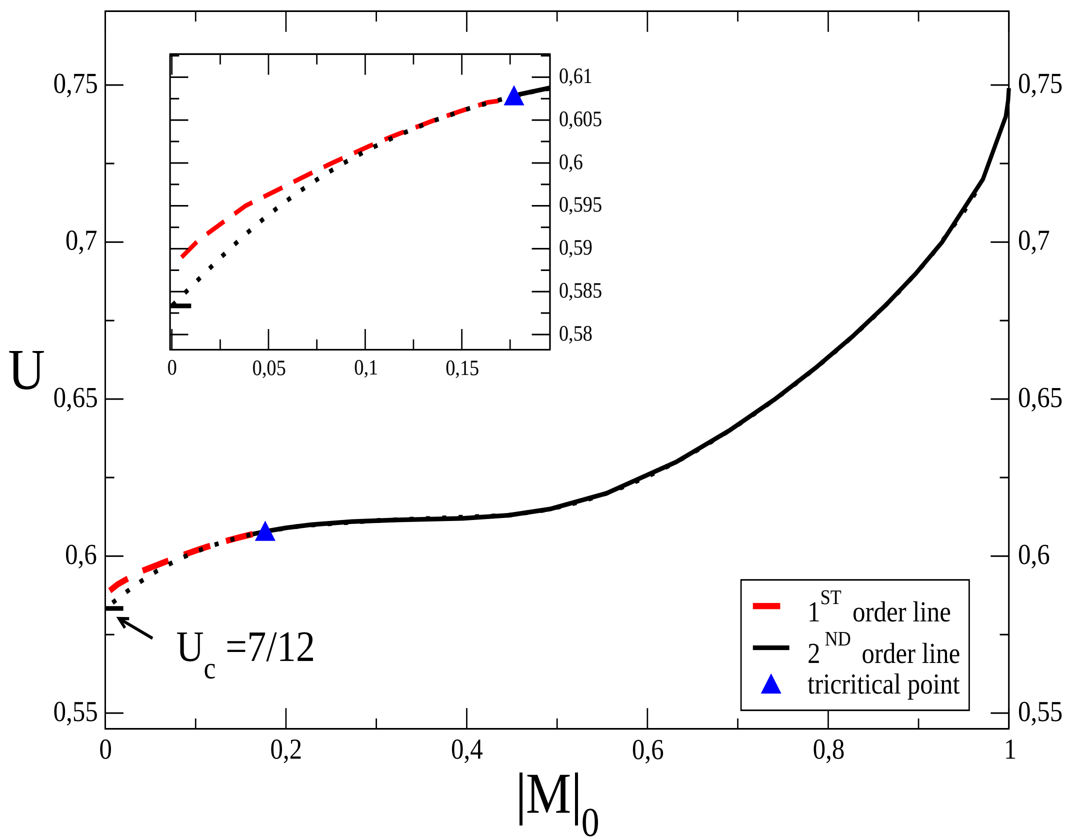
# Velocity PDF



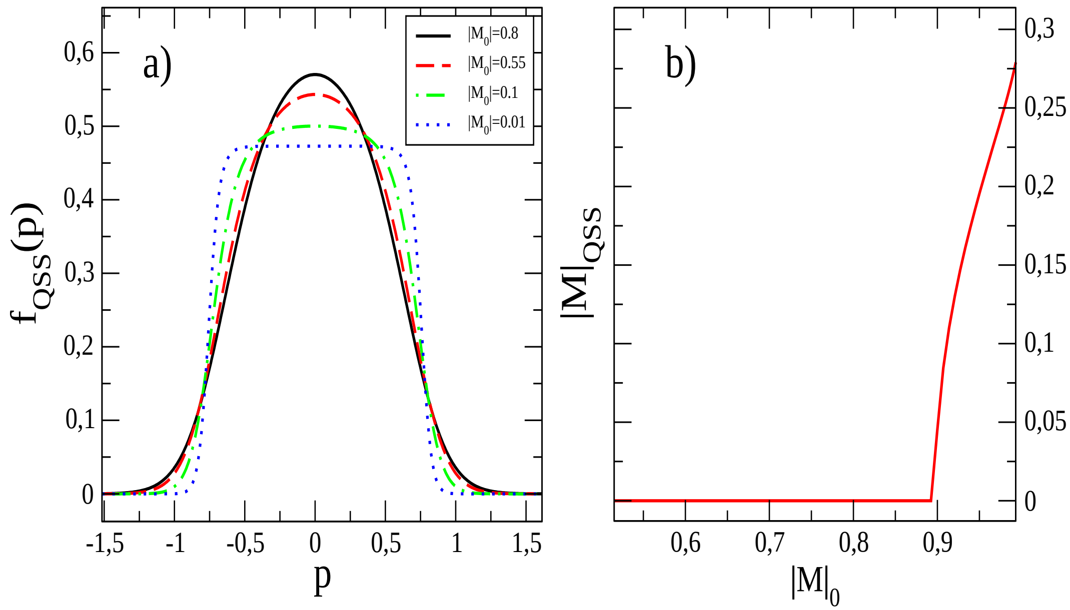

$$|M_0| = \sin(\Delta\theta_0)/\Delta\theta_0$$

Non Gaussian velocity distributions

# Tricritical point



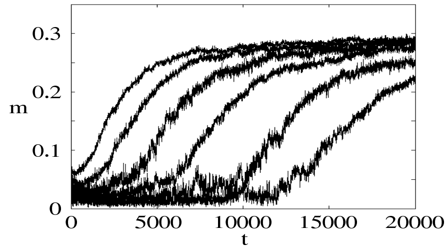
# Features of the phase transition

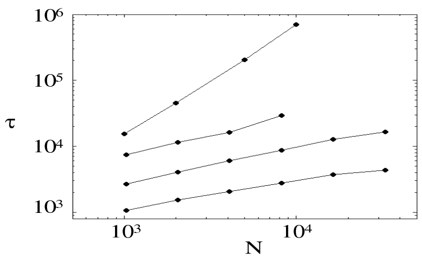


# Addition of a short range contribution

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} [1 - \cos(\theta_i - \theta_j)] - K \sum_{i=1}^{N} \cos(\theta_{i+1} - \theta_i) \ .$$

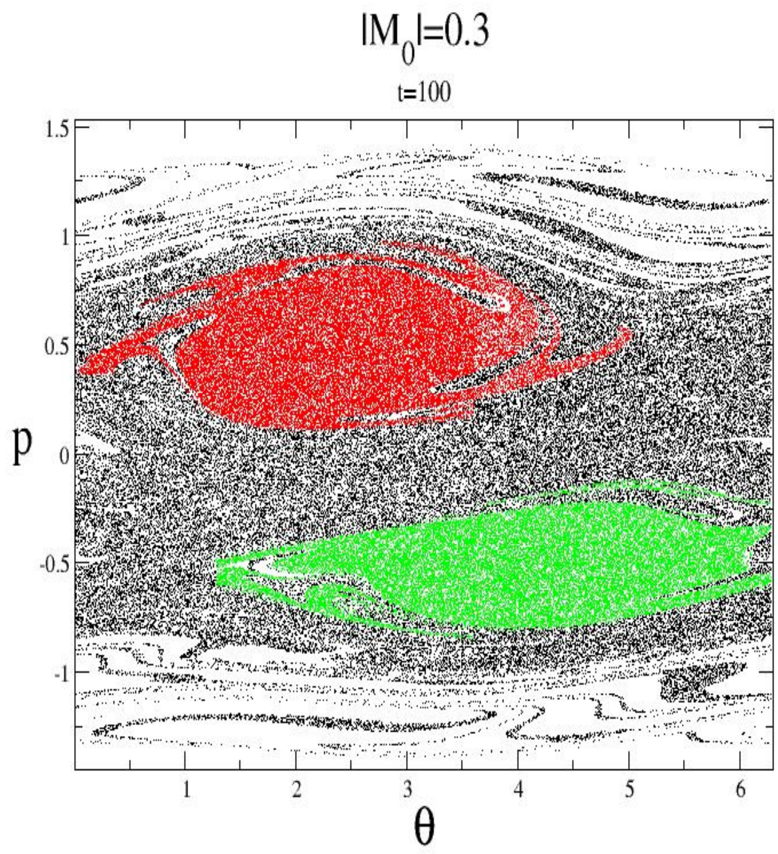
LEFT: $M(t)$ vs. $t$; $K = 0.05$, $U = 0.71$, $N = 1024, \ldots, 32768$
RIGHT: increase of the lifetime; $K = 0, 0.0025, 0.05, 0.1$



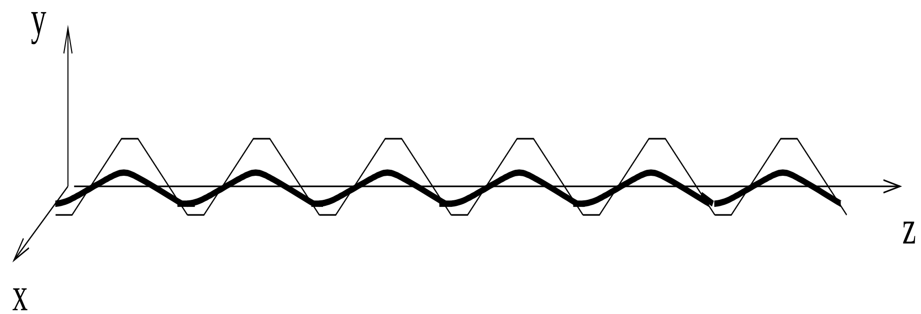



# HMF core-halo structure

Refinements of maximum entropy methods should take into account the "true" dynamics.



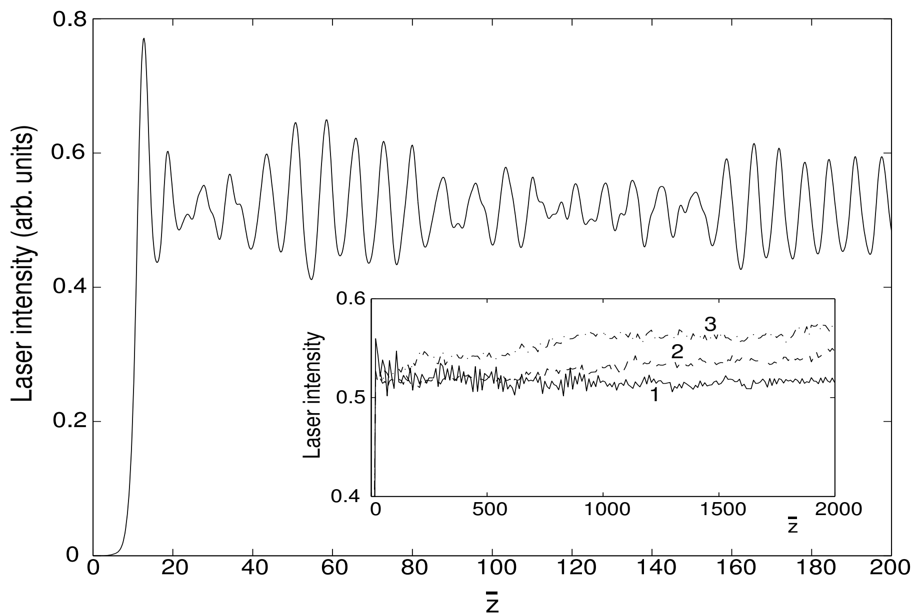
# Free Electron Laser




Colson-Bonifacio model

$$
\begin{aligned}
\frac{d\theta_j}{dz} &= p_j \\
\frac{dp_j}{dz} &= -\mathbf{A}e^{i\theta_j} - \mathbf{A}^* e^{-i\theta_j} \\
\frac{d\mathbf{A}}{dz} &= i\delta\mathbf{A} + \frac{1}{N}\sum_j e^{-i\theta_j}
\end{aligned}
$$

# Quasi-stationary states




$N = 5000$ (curve 1), $N = 400$ (curve 2), $N = 100$ (curve 3)

On a first stage the system converges to a quasi-stationary state. Later it relaxes to Boltzmann-Gibbs equilibrium on a time $O(N)$. The quasi-stationary state is a Vlasov equilibrium, sufficiently well described by Lynden-Bell's Fermi-like distributions.

# Vlasov equation

In the $N \to \infty$ limit, the single particle distribution function $f(\theta, p, t)$ obeys a Vlasov equation.

$$\begin{aligned}
\frac{\partial f}{\partial z} &= -p\frac{\partial f}{\partial \theta} + 2(A_x \cos\theta - A_y \sin\theta)\frac{\partial f}{\partial p} \quad , \\
\frac{\partial A_x}{\partial z} &= -\delta A_y + \frac{1}{2\pi}\int f \cos\theta \, d\theta dp \quad , \\
\frac{\partial A_y}{\partial \bar{z}} &= \delta A_x - \frac{1}{2\pi}\int f \sin\theta \, d\theta dp \; .
\end{aligned}$$

with $\mathbf{A} = A_x + iA_y = \sqrt{I}\exp(-i\varphi)$

# Vlasov equilibria

Lynden-Bell entropy maximization

$$S_{LB}(\bar{f}) = -\int dp d\theta \left( \frac{\bar{f}}{f_0} \ln \frac{\bar{f}}{f_0} + \left(1 - \frac{\bar{f}}{f_0}\right) \ln \left(1 - \frac{\bar{f}}{f_0}\right)\right).$$

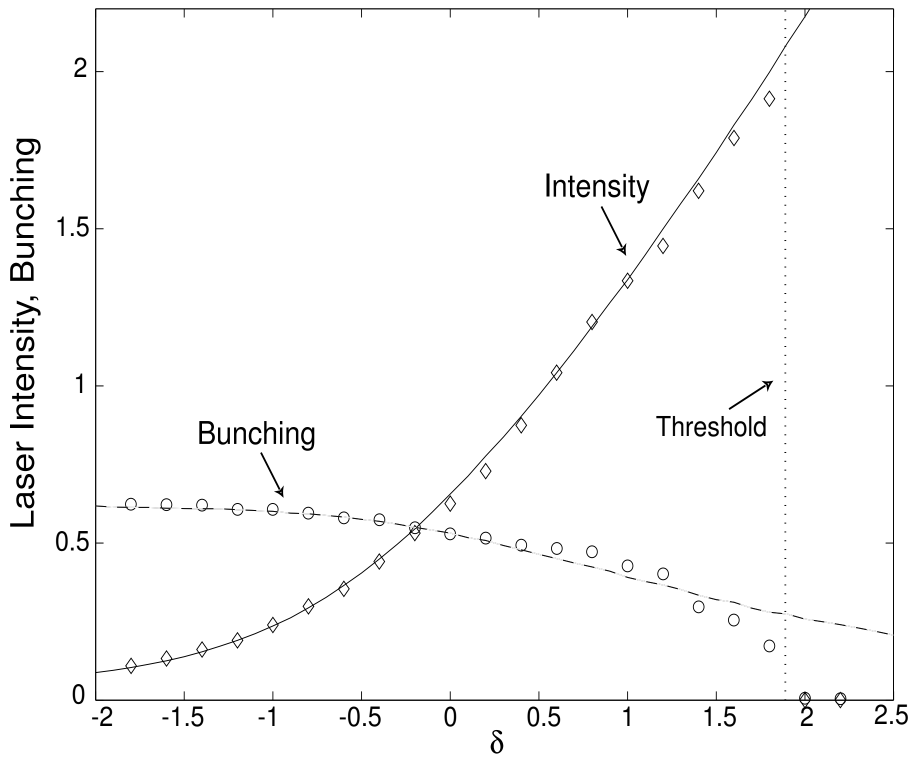
$$S_{LB}(\varepsilon, \sigma) = \max_{\bar{f}, A_x, A_y} [S_{LB}(\bar{f}) | H(\bar{f}, A_x, A_y) = N\varepsilon; \int d\theta dp \bar{f} = 1; P(\bar{f}, A_x, A_y) = \sigma].$$

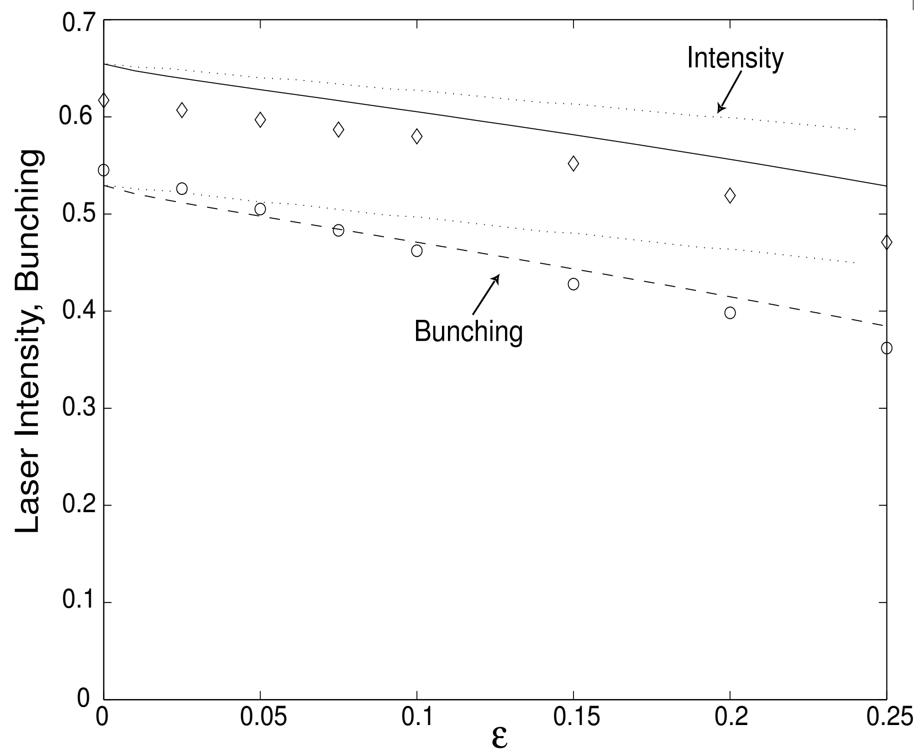
$$\bar{f} = f_0 \frac{e^{-\beta(p^2/2 + 2A\sin\theta) - \lambda p - \mu}}{1 + e^{-\beta(p^2/2 + 2A\sin\theta) - \lambda p - \mu}} \;.$$

Non-equilibrium field amplitude

$$A = \sqrt{A_x^2 + A_y^2} = \frac{\beta}{\beta\delta - \lambda} \int dp d\theta \, \sin\theta \bar{f}(\theta, p).$$

# Results


Laser Intensity, Bunching
Intensity
Bunching
Threshold
δ


Laser Intensity, Bunching
Intensity
Bunching
ε

# Conclusions

- Mean-field Hamiltonian systems with many degrees of freedom display interesting statistical and dynamical properties.
- Non equilibrium quasi-stationary states arise “naturally" from water-bag initial conditions. Their life-time increases with system size.
- Vlasov (non collisional) equation correctly describes the “initial" dynamics of mean-field Hamiltonians.
- Lynden-Bell maximum entropy principle provides a theoretical approach to quasi-stationary states.
- Collective phenomena of wave-particle interactions (free electron laser) are the result of a Lynden-Bell maximum entropy principle.

# References


- T. Dauxois, S. Ruffo, E. Arimondo and M. Wilkens (Eds.), *Dynamics and statistics of systems with long-range interactions*, Springer Lecture Notes in Physics **602** (2002).
- Y.Y. Yamaguchi, J. Barré, F. Bouchet, T. Dauxois, S. Ruffo, *Stability criteria of the Vlasov equation and quasi-stationary states of the HMF model*, Physica A, **337**, 36 (2004).
- J. Barré, T. Dauxois, G. De Ninno, D. Fanelli, S. Ruffo:*Statistical theory of high-gain free-electron laser saturation*, Phys. Rev. E, Rapid Comm., **69**, 045501 (R) (2004).
- M. Antoni, A. Torcini and S. Ruffo *First-order microcanonical transitions in finite mean-field models*, Europhysics Letters, **66**, 645 (2004).
- D. Mukamel, S. Ruffo and N. Schreiber, *Breaking of ergodicity and long relaxation times in systems with long-range interactions*, Phys. Rev. Lett., **95**, 240604 (2005).
- A. Campa, A. Giansanti, D. Mukamel and S. Ruffo, *Dynamics and thermodynamics of rotators interacting with both long and short range couplings* , Physica A, **365**, 177 (2006)
- A. Antoniazzi, D. Fanelli, J. Barré, P.-H. Chavanis, T. Dauxois and S. Ruffo, *A maximum entropy principle explains quasi-stationary states in systems with long range interactions: the exemple of the HMF model*, preprint (2006)