QCD@Work 2007 - Martina Franca

The cusp anomaly, integrability and AdS/CFT

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Plan of the talk:

- The cusp anomaly: a very important quantity in QCD and gauge theories
- Old and new results in integrability
- Strong coupling and string theory

The cusp anomaly appears everywhere...

- Renormalization of Wilson loops
- Infrared divergences in perturbative QCD and HQET
- Scattering of gluons (planar limit)

Renormalization of Wilson loops

Gauge field

$$W(C) = \langle 0 | P e^{ig \int_C dx_\mu A^\mu} | 0 \rangle$$

This quantity is UV divergent

For example in QED, after a gaussian integral

$$W(C) = e^{-\frac{e^2}{2} \int dx_\mu dy_\nu \frac{g_{\mu\nu}}{(x-y)^2 + a^2}}$$

point splitting regulator

Renormalization of Wilson loops

in QED

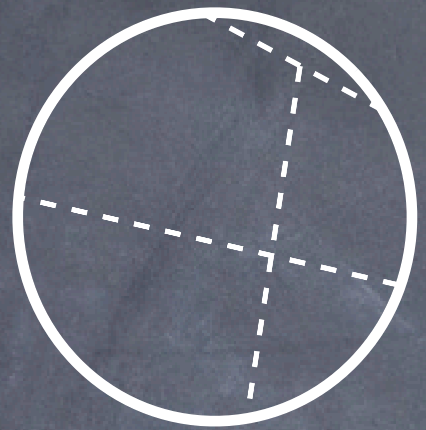
$$W(C) = e^{-\frac{e^2}{2}\int dx_\mu dy_\nu \frac{g_{\mu\nu}}{(x-y)^2+a^2}}$$

Notice that

$$\int dx_\mu dy_\nu \frac{g_{\mu\nu}}{(x-y)^2+a^2} \sim \frac{L(C)}{a} + \text{finite}$$

The divergence structure is the same in all gauge theories (including QCD and SUSY theories)

Renormalization of Wilson loops



in a generic gauge theory

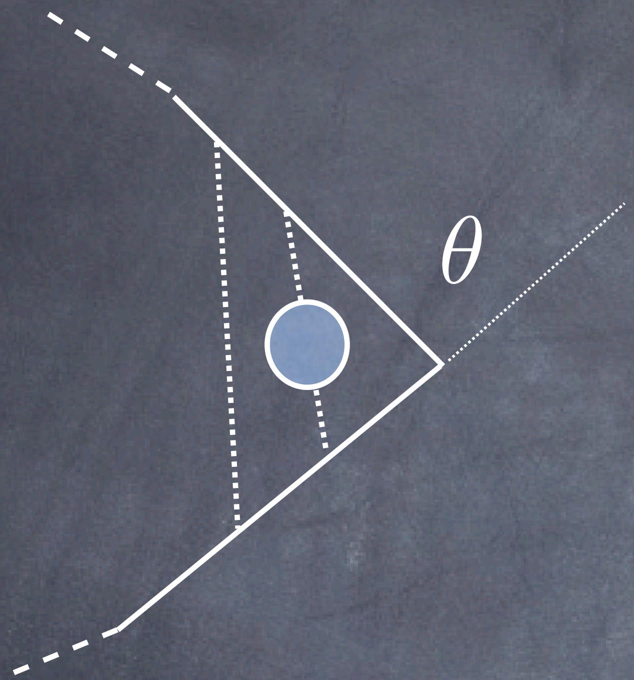
Multiplicative renormalization

$$W(C) = \langle 0 | P e^{ig \int_C dx_\mu A^\mu} | 0 \rangle \sim Z_C f(g_{\rm ren}, C)$$

$$Z_C = e^{-\gamma L/a}$$

This renormalization is enough (Polyakov) unless...

Renormalization of Wilson loops (with cusps)



$$\ln Z_C \simeq \gamma L/a + \Gamma_{\text{cusp}}(g,\theta)\ln 1/a + \text{finite}$$

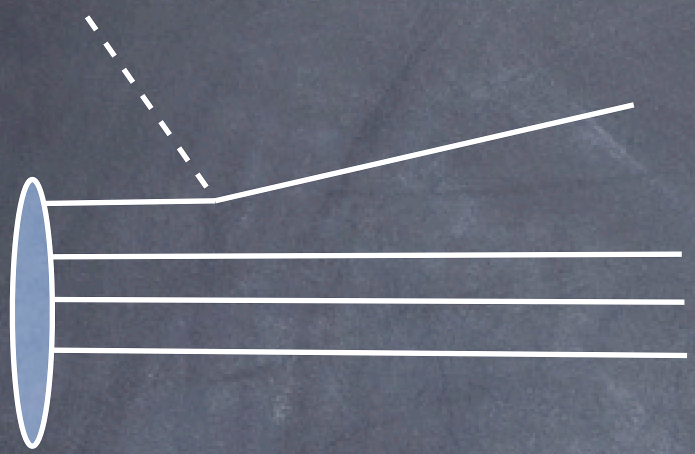
the cusp anomalous dimension
(aka cusp anomaly)

$$\Gamma_{\text{cusp}}(g,\theta) \simeq \frac{\alpha_S N_c}{2\pi}(\theta\cot\theta - 1) + \mathcal{O}(\alpha_S^2 N_c^2)$$

$$\Gamma_{\text{cusp}}(g,\theta) \to f(g)\theta$$

when $\theta \to \infty$ in Minkovsky space

Who cares about the cusp anomaly?
The Altarelli-Parisi equation



Parton picture of the hadrons

$$q(x, \ln Q^2/Q_0^2)$$

Parton distribution function

$$\frac{dq(x,t)}{dt} = \frac{\lambda(t)}{2\pi} \int_x^1 \frac{dy}{y} q(y,t) P_{qq}(y)$$

Evolution

Coupling constant

Splitting function

Who cares about the cusp anomaly? The Altarelli-Parisi equation

$$P_{qq}(z) = a\delta(1-z) + \frac{f(g)}{(1-z)_+} + \ldots$$

conjecture

where

$$\int_0^1 dz A(z) \frac{1}{(1-z)_+} \equiv \int dz \frac{A(z) - A(1)}{1-z}$$

So this is a measurable quantity which defines the behavior of soft gluons (Korchemsky, Radyushkin, Marchesini, etc.)

It also determines the behavior of MHV scattering of gluons (in the planar limit) (Sterman, Bern, Kosover, Dixon, Alday and Maldacena)

How do we calculate the $f(g)$?

- Perturbation series in g (shut up and calculate approach)
- or ... any better idea?

Integrability in planar, N=4 SYM

- N=4 is the maximally supersymmetric gauge theory in 4 dimensions
- It is a conformal theory
- Perturbatively it looks a lot like QCD although the spectrum is quite different
- It is most probably INTEGRABLE (namely it looks a lot like the Heisenberg model or any other integrable model you have in mind)

Cusp anomaly in N=4 SYM

The cusp anomaly, its interpretation and properties extend smoothly to N=4 SYM

Understanding them in N=4 can lead to a deeper understanding of the same quantities in QCD (unlike for example the spectrum and other low-energy observables)

Leading twist long operators

Take some colored scalar field Φ

$\mathcal{O}_L = \mathrm{Tr}(\Phi D^L \Phi)$ where $L \gg 1$

Under operator mixing it mixes with

$$\mathcal{O}_{L,m} = Tr(D^m \Phi D^{L-m} \Phi)$$

By diagonalizing the mixing matrix we find

$$\Delta = L + f(g) \log L + ...$$

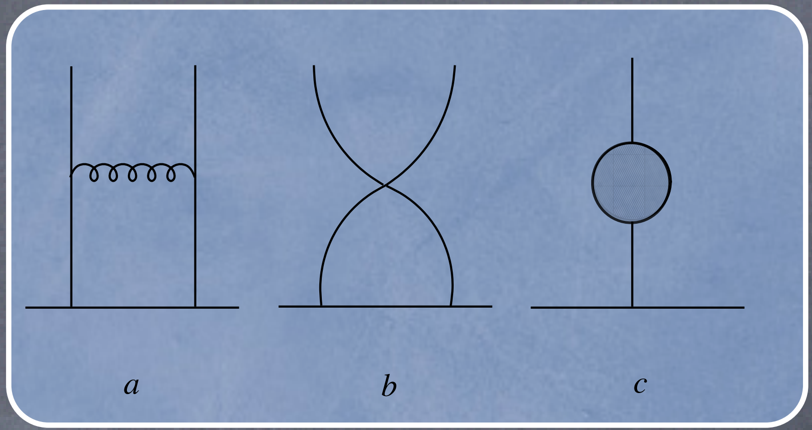
again the cusp anomaly!

Spin chains and long operators

Operators and states

$$X^L \equiv |\downarrow, \downarrow, \ldots, \downarrow\rangle$$

$$X^m Y X^{L-m-1} \equiv |\downarrow, \downarrow, \ldots, \uparrow, \downarrow, \ldots, \downarrow\rangle$$



Find the hamiltonian which reproduces mixing at a given order in a loop expansion

Find the eigenvalues of this hamiltonian

This gives the anomalous dimensions

(Minahan, Zarembo, Beisert, Eden, Staudacher and many others)

Spin chains and long operators

However this is cumbersome

Alternative way:

1) find the properties of elementary excitation (dispersion relation)

2) find their interaction (assuming integrability)

3) solve the Bethe-ansatz equations

Analytic expression for f(g)

Define

$$s(t) = \frac{e^t - 1}{t}\sigma(t)$$

solve

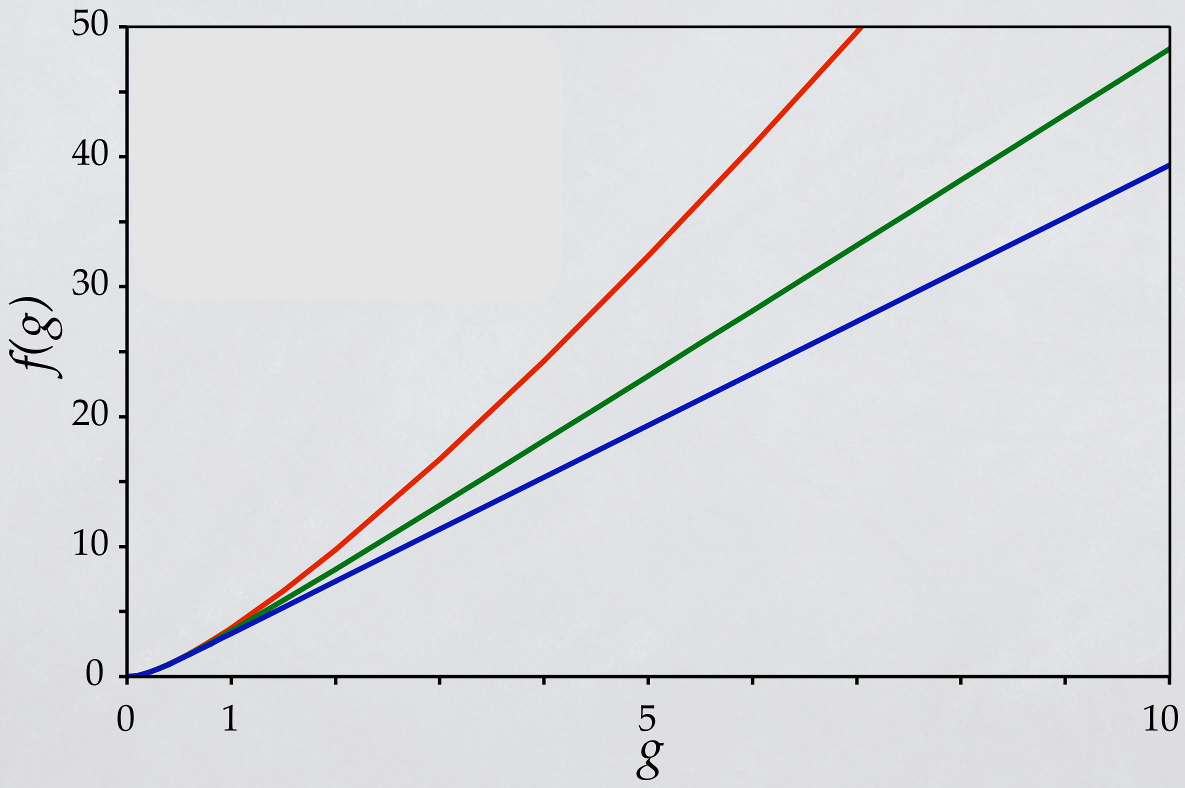
$$s(t) = K(2gt, 0) - 4g^2 \int_0^\infty dt' K(2gt, 2gt') \frac{t'}{e^{t'} - 1} s(t')$$

The cusp anomalous dimension is given by

$$f(g) = 16g^2\sigma(0)$$

Analytic (numerical) expression for f(g)

(Benna, Benvenuti, Klebanov, A.S.)



Strong coupling limit

From the numerical solution one extracts the strong coupling limit of f(g)

$$f(g) = (4 \pm 10^{-6})g + (0.661907 \pm 2\ 10^{-6}) - \frac{0.0232 \pm 10^{-4}}{g} + \dots$$

while from AdS/CFT we get

$$f(g) = 4g + 3\log 2/\pi + \dots$$

and many other properties of f(g) (like the analytic structure in the complex g plane)

This is remarkable!

- We have essentially solved the planar limit of a gauge theory
- We can obtain the most important function(s) describing properties of the theory for arbitrary coupling (weak to strong)
- We can test highly non-trivial predictions of AdS/CFT
- We can think of applying the same techniques to other theories (the planar limit maybe more important than the conformal invariance)

Weak coupling expansion of f(g)

$$
\begin{aligned}
f(g) = 8g^2 - \frac{8}{3}\,\pi^2 g^4 + \frac{88}{45}\,\pi^4 g^6 - 16\left(\frac{73}{630}\,\pi^6 + 4\,\zeta(3)^2\right) g^8 \\
+ 32\left(\frac{887}{14175}\,\pi^8 + \frac{4}{3}\,\pi^2\zeta(3)^2 + 40\,\zeta(3)\,\zeta(5)\right) g^{10} \\
- 64\left(\frac{136883}{3742200}\,\pi^{10} + \frac{8}{15}\,\pi^4\zeta(3)^2 + \frac{40}{3}\,\pi^2\zeta(3)\,\zeta(5) \right. \\
\left. + 210\,\zeta(3)\,\zeta(7) + 102\,\zeta(5)^2\right) g^{12} \\
+ 128\left(\frac{7680089}{340540200}\,\pi^{12} + \frac{47}{189}\,\pi^6\zeta(3)^2 + \frac{82}{15}\,\pi^4\zeta(3)\,\zeta(5) + 70\pi^2\zeta(3)\,\zeta(7) \right. \\
\left. + 34\pi^2\zeta(5)^2 + 1176\,\zeta(3)\,\zeta(9) + 1092\,\zeta(5)\,\zeta(7) + 4\,\zeta(3)^4\right) g^{14}
\end{aligned}
$$

Details on the equation

$$s(t) = K(2gt, 0) - 4g^2 \int_0^\infty dt' K(2gt, 2gt') \frac{t'}{e^{t'} - 1} s(t'), \tag{6}$$

with the kernel given by [21]

$$K(t, t') = K^{(m)}(t, t') + 2K^{(c)}(t, t'). \tag{7}$$

The main scattering kernel $K^{(m)}$ of [18] is

$$K^{(m)}(t, t') = \frac{J_1(t)J_0(t') - J_0(t)J_1(t')}{t - t'}, \tag{8}$$

and the dressing kernel $K^{(c)}$ is defined as the convolution

$$K^{(c)}(t, t') = 4g^2 \int_0^\infty dt'' K_1(t, 2gt'') \frac{t''}{e^{t''} - 1} K_0(2gt'', t'), \tag{9}$$

where K_0 and K_1 denote the parts of the kernel that are even and odd, respectively, under change of sign of t and t':

$$\begin{aligned} K_0(t, t') &= \frac{tJ_1(t)J_0(t') - t'J_0(t)J_1(t')}{t^2 - t'^2} \\ &= \frac{2}{tt'} \sum_{n=1}^{\infty} (2n - 1)J_{2n-1}(t)J_{2n-1}(t'), \end{aligned} \tag{10}$$

$$\begin{aligned} K_1(t, t') &= \frac{t'J_1(t)J_0(t') - tJ_0(t)J_1(t')}{t^2 - t'^2} \\ &= \frac{2}{tt'} \sum_{n=1}^{\infty} (2n)J_{2n}(t)J_{2n}(t'). \end{aligned} \tag{11}$$