Infrared non-perturbative QCD running coupling from Bogolubov approach

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We start with QCD Lagrangian with three massless quarks $(u,\, d,\, s)$ with number of colours $N = 3$

$$L \,=\, \sum_{k=1}^{3}(\frac{\imath}{2}(\bar\psi_k\gamma_\mu\partial_\mu\psi_k \,-\partial_\mu\bar\psi_k\gamma_\mu\psi_k\,)+\, g\bar\psi_k\gamma_\mu t^a A_\mu^a\psi_k)$$
$$-\,\frac{1}{4}\,(F^a_{\mu\nu}F^a_{\mu\nu}); \tag{1}$$
$$F^a_{\mu\nu}\,=\,\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\,+g\,f_{abc}A^b_\mu A^c_\nu\,.$$

Bogolubov approach: N.N. Bogolubov. Physica Suppl., **26**, 1 (1960); N.N. Bogolubov, *Quasi-averages in problems of statistical mechanics.* Preprint JINR D-781, (Dubna: JINR, 1961).

Application to QFT: B.A.A., Theor. Math. Phys., **140**, 1205 (2004).
Application to QCD: B.A.A., Phys. Atom. Nucl., **69**, 1588 (2006); B.A.A., M.K. Volkov and I.V. Zaitsev, Int. Journ. Mod. Phys. A, **21**, 5721 (2006).
Results of calculation of hadron low-energy parameters: m_π, f_π, m_σ,
Γ_σ, $< \bar{q}q >$ are quite consistent.

Bogolubov procedure **add − subtract**.

$$L = L_0 + L_{int};$$

$$L_0 = \frac{\imath}{2}(\bar{\psi}\gamma_\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma_\mu\psi) - \frac{1}{4}F^a_{0\,\mu\nu}F^a_{0\,\mu\nu} +$$
$$+\frac{G}{3!}\cdot\ f_{abc}\,F^a_{\mu\nu}\,F^b_{\nu\rho}\,F^c_{\rho\mu}\,; \tag{2}$$

$$L_{int} = g\,\bar{\psi}\gamma_\mu t^a A^a_\mu\psi - \frac{1}{4}(F^a_{\mu\nu}F^a_{\mu\nu} - F^a_{0\,\mu\nu}F^a_{0\,\mu\nu}) -$$
$$-\frac{G}{3!}\cdot\ f_{abc}\,F^a_{\mu\nu}\,F^b_{\nu\rho}\,F^c_{\rho\mu}\,. \tag{3}$$

Here $F_{0\,\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\frac{G}{3!}\cdot\ f_{abc}\,F^a_{\mu\nu}\,F^b_{\nu\rho}\,F^c_{\rho\mu}$ means non-local vertex in the momentum space

$$(2\pi)^4\,G\ f_{abc}\,(g_{\mu\nu}(q_\rho pk - p_\rho qk) + g_{\nu\rho}(k_\mu pq - q_\mu pk) +$$
$$g_{\rho\mu}(p_\nu qk - k_\nu pq) + q_\mu k_\nu p_\rho - k_\mu p_\nu q_\rho) \times$$
$$\times F(p,q,k)\,\delta(p+q+k); \tag{4}$$

$F(p,q,k)$ is a form-factor and $p,\mu;\ q,\nu;\ k,\rho$ – incoming momenta and Lorentz indices of gluons (four-gluon, five-gluon and six-gluon vertices are present also).

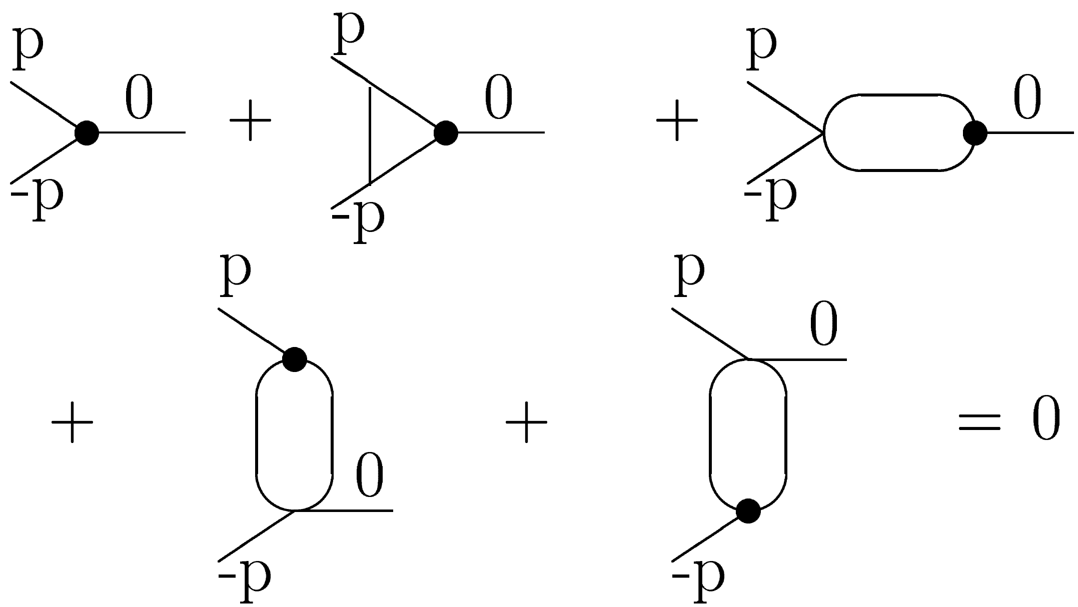
(2) – new **free** Lagrangian L_0, (3) – new **interaction** Lagrangian L_{int}. Then compensation conditions will consist in demand of full connected tree-gluon vertices, following from Lagrangian L_0, to be zero. This demand gives a non-linear equation for form-factor F.

These equations according to terminology of works by Bogolubov are called **compensation equations**. In a study of these equations it is always evident the existence of a perturbative trivial solution (in our case $G = 0$), but, in general, a non-perturbative non-trivial solution may also exist. Just the quest of a non-trivial solution inspires the main interest in such problems. The goal of a study is a quest of an adequate approach, the first non-perturbative approximation of which describes the main features of the problem.

1) In compensation equation we restrict ourselves by terms with loop numbers 0, 1.
2) In expressions thus obtained we perform a procedure of linearizing, which leads to linear integral equations.
3) We integrate by angular variables of the 4-dimensional Euclidean space.
4) We look for a solution with the following simple dependence on all three variables

$$F(p_1,\, p_2,\, p_3) \,=\, F\Big(\frac{p_1^2\,+\,p_2^2\,+\,p_3^2}{2}\Big)\,; \tag{5}$$

We plan to take into account corrections to the first approximation in the forthcoming study.



$$F(x) = -\frac{G^2N}{64\pi^2}\Big(\int_0^Y F(y)y\,dy - \frac{1}{12x^2}\int_0^x F(y)y^3dy +$$
$$+\frac{1}{6x}\int_0^x F(y)y^2dy + \frac{x}{6}\int_x^Y F(y)dy - \frac{x^2}{12}\int_x^Y \frac{F(y)}{y}dy\Big) +$$

$$+\frac{GgN}{16\pi^2}\int_0^Y F(y)dy+\frac{GgN}{24\pi^2}(\int_{x/4}^Y \frac{(5x-24y)}{(x-8y)}F(y)dy+$$
$$+\int_{3x/16}^{x/4}\frac{(3x-16y)^2(3x-8y)}{x^2(x-8y)}F(y)\,dy)\,. \quad (6)$$

Here $x = p^2$ and $y = q^2$, where q is an integration momentum. We introduce here an effective cut-off Y, which limited an infrared region and consider the equation at interval $[0,\,Y]$ under condition

$$F(Y) = 0\,; \quad G = 0 \;\; for\; p^2 > Y\,. \quad (7)$$

We shall solve equation (6) by iterations. That is we expand the last lines of (6) in powers of x and take at first only constant term.

$$F_0(x) = -\frac{G^2N}{64\pi^2}\Big(\int_0^Y F_0(y)ydy - \frac{1}{12x^2}\int_0^x F_0(y)y^3dy +$$
$$+\frac{1}{6x}\int_0^x F_0(y)y^2dy + \frac{x}{6}\int_x^Y F_0(y)dy -$$
$$-\frac{x^2}{12}\int_x^Y \frac{F_0(y)}{y}dy\Big) + \frac{3\,G\,g\,N}{16\,\pi^2}\int_0^Y F_0(y)\,dy\,. \tag{8}$$

$$F_0(z) = \frac{1}{2}\Big(-\frac{G^2N}{64\pi^2}\int_0^Y F(y)ydy + \frac{3GgN}{16\pi^2}\int_0^Y F(y)dy\Big) \times$$
$$\times G^{31}_{15}\left(z\,\middle|^{0}_{1,\,1/2,\,0,\,-1/2,\,-1}\right) + \tag{9}$$
$$+C_1\,G^{10}_{04}\left(z\,\middle|\frac{1}{2},1,-\frac{1}{2},-1\right) + C_2G^{10}_{04}\left(z\,\middle|1,\frac{1}{2},-\frac{1}{2},-1\right);$$
$$z\; = \;\frac{G^2\,N\,x^2}{1024\,\pi^2}\,; \quad z_0\; = \;\frac{G^2\,N\,Y^2}{1024\,\pi^2}\,.$$

Here

$$G_{pq}^{mn}(z\,|^{a_1,..,a_p}_{b_1,..,b_q})\,; \tag{10}$$

is a Meijer function.
Boundary conditions

$$[2\,z^2\frac{d^3\,F_0(z)}{dz^3}\;+9\,z\,\frac{d^2\,F_0(z)}{dz^2}\;+\;\frac{d\,F_0(z)}{dz}]_{z\,=\,z_0}=0\,;$$
$$[2\,z^2\,\frac{d^2\,F_0(z)}{dz^2}\;+5\,z\,\frac{d\,F_0(z)}{dz}\;+\;F_0(z)]_{z\,=\,z_0}=1\,; \tag{11}$$

Conditions (7, 11) fulfil for $z_0 = \infty$, $C_1 = 0$, $C2 = 0$. However with these parameters the first integral in (8) diverges and we have no consistent solution. As the next step we take into account terms proportional to $\sqrt{z}$.

$$F(z) \;=\; 1 \;+\; \frac{g\sqrt{z}}{4\,\sqrt{3}\,\pi}(\ln(z)\;-\;4\,\gamma\;-11\;\ln(2)\;+\;\frac{7}{12})\;+$$

$$+\frac{2}{3\,z}\int_0^z F(t)\,t\,dt\;-\;\frac{4}{3\,\sqrt{z}}\int_0^z F(t)\,\sqrt{t}dt\;-$$
$$-\frac{4\,\sqrt{z}}{3}\int_z^{z_0} F(t)\,\frac{dt}{\sqrt{t}}\;+\;\frac{2\,z}{3}\int_z^{z_0} F(t)\,\frac{dt}{t}\,; \tag{12}$$

$$F(z)=\frac{1}{2}G^{31}_{15}(z|^{0}_{1,1/2,0,-1/2,-1})\,- \tag{13}$$
$$-\frac{g\sqrt{3}}{16\pi}\,G^{31}_{15}(z|^{1/2}_{1,1/2,1/2,-1/2,-1})+$$
$$+C_1G^{10}_{04}(z|1/2,1,-1/2,-1)+$$
$$+C_2G^{10}_{04}(z|1,1/2,-1/2,-1);$$

$$
\begin{aligned}
&\frac{1}{2}G^{30}_{04}(z|\frac{1}{2},0,-\frac{1}{2},0)-\frac{g\sqrt{3}}{16\pi}(\frac{1}{2}G^{30}_{04}(z|\frac{1}{2},0,0,-\frac{3}{2})+\\
&+G^{31}_{15}(z|^{-1}_{1/2,0,0,0,-3/2}))+C_1G^{10}_{04}(z|0,\frac{1}{2},0,-\frac{1}{2})+\\
&+C_2\,G^{10}_{04}(z|1/2,0,0,-1/2)- \\
&-\frac{1}{2\sqrt{z}}-\frac{g\sqrt{3}}{4\pi}(\ln(z)+\frac{9}{4}-4\gamma-11\ \ln(2))\ =\ 0\,;\\
&\frac{1}{2}\,G^{30}_{04}(z|\frac{1}{2},0,-\frac{1}{2},1)-\frac{g\sqrt{3}}{16\pi}(\frac{1}{2}\,G^{31}_{15}(z|^{0}_{1/2,0,0,1,-3/2})+\\
&+G^{31}_{15}(z|^{-2}_{1/2,0,0,0,-3/2})-2G^{31}_{15}(z|^{-1}_{1/2,0,0,0,-3/2}))+\\
&+C_1\,G^{10}_{04}(z|0,1,\frac{1}{2},-\frac{1}{2})+C_2G^{10}_{04}(z|\frac{1}{2},1,0,-\frac{1}{2})+\\
&+\frac{1}{4\sqrt{z}}-\frac{g\sqrt{3}}{4\pi}=0\,;\qquad z=z_0\,;
\end{aligned}
\tag{14}
$$

Condition $F(0) = 1$ leads to the following relation

$$1 + \frac{G^2 N}{64\pi^2} \int_0^Y F_0(y) y dy = \frac{3GgN}{16\pi^2} \int_0^Y F_0(y) dy; \quad (15)$$

We have also condition

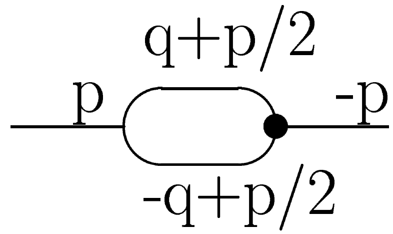
$$F(z_0) \;=\; 0\,; \quad (16)$$

The conditions define values of parameters

$$g(Y) \;=\; 2.55779\,; \quad z_0 \;=\; \frac{G^2\,N\,Y^2}{1024\,\pi^2} \;=\; 1.915838\,;$$
$$C_1 \;=\; 0.06172743\,; \quad C_2 \;=\; -0.1640803\,. \quad (17)$$

We would draw attention to the fixed value of parameter z_0. The solution exists only for this value and it plays the role of eigenvalue. As a matter of fact from the beginning the existence of such eigenvalue is by no means evident.

We use Schwinger-Dyson equation for gluon polarization operator to obtain a contribution of additional effective vertex to the running QCD coupling constant α_s.



$$\Delta\Pi_{\mu\nu}(x) \;=\; \frac{g\,G\,N}{2\,(2\,\pi)^4}\int \Gamma^0_{\mu\rho\sigma}(p,-q-\frac{p}{2},q-\frac{p}{2})\times$$
$$\times\frac{\Gamma^{eff}_{\nu\rho\sigma}(-p,q+\frac{p}{2},-q+\frac{p}{2})F(q^2+\frac{3p^2}{4})dq}{(q^2+p^2/4)^2\;-\;(pq)^2}\,. \qquad (18)$$

After angular integration

$$\Delta\Pi_{\mu\nu}(p) = (g_{\mu\nu}p^2 - p_\mu p_\nu)\Pi(x);\; x = p^2;\; y' = q^2 + \frac{3x}{4};$$

$$\Pi(x) \;=\; -\,\frac{g\,G\,N}{32\,\pi^2}\times \tag{19}$$
$$\times\Big(\frac{1}{x^2}\int_{3x/4}^{x}\frac{F(y')dy'}{y'-x/2}\,(16\frac{y'^3}{x^2}-48\frac{y'^2}{x}+45y-\frac{27}{2}x)\;+$$
$$+\;\int_{x}^{Y}\frac{F(y')dy'}{y'-x/2}\,(-\,3y'\;+\;\frac{5}{2}\,x)\Big)\,.$$

So we have modified one-loop expression for $\alpha_s(p^2)$

$$\alpha_s(x) = \frac{4\,\pi\,\alpha_s(p_0^2)}{4\pi + b_0\alpha_s(p_0^2)\ln(x/\Lambda^2) + 4\pi\Pi(x)}; \tag{20}$$
$$x = p^2\,; \quad b_0 \;=\; 11 - \frac{2\,N_f}{3}\,.$$

It is remarkable that function $\Pi(x)$ at $x \;=\; Y$ $(z = z_0)$ turns to be very small, almost zero. We just expect this quantity to be zero exactly. So this property of the approximated polarization operator indicates the consistency of the procedure being used. Namely we normalize $\alpha_s(p^2)$ at point p_0, which correspond to our cut-off Y. Performing the well-known transformations in expression (20) we have for $u < u_0$

$$\alpha_s(u) \;=\; \frac{4\,\pi}{9}(\ln(u)\;+\;\frac{8\,\sqrt{N}\,\pi}{9\,g}\,I)^{-1}\,; \qquad (21)$$

$$I = -\int_{\frac{z}{16}}^{z}\frac{F_0(t)dt}{\sqrt{t}(2\sqrt{t}-\sqrt{z})}[16\frac{t^{3/2}}{z}-48\frac{t}{\sqrt{z}}+45\sqrt{t}-$$

$$-\frac{27}{2}\sqrt{z}]+\int_{z}^{z_0}\frac{F_0(t)dt}{\sqrt{t}(2\sqrt{t}-\sqrt{z})}[-3\sqrt{t}+\frac{5}{2}\sqrt{z}];$$

$$z = \frac{G^2 N x^2}{1024\pi^2}\,;\;\; u \;=\; \frac{x}{\Lambda^2_{QCD}}\,;\;\; u_0 = \frac{Y}{\Lambda^2_{QCD}} = 14.6133\,.$$

For $u > u_0$ we use the perturbative one-loop expression

$$\alpha_s(u) = \frac{4\,\pi}{b_0\,\ln(u)}\,. \tag{22}$$

The self-consistent result for expressions (21, 22) with account of previous relations is unique and reads as follows

$$\begin{aligned} z_0 &= 1.91584\,; \quad u_0 = 14.6133\,; \quad \alpha_0 = 0.52062\,; \\ C_1 &= 0.06172743\,; \quad C_2 = -\,0.1640803\,. \end{aligned} \tag{23}$$

Behavior of α_s (21) with $u = Q^2/\Lambda^2_{QCD}$ for $\Lambda_{QCD} = 0.2\,GeV$ and $0.05\,GeV < Q < 1\,GeV$ is presented at Fig. 3.

The behaviour with maximum at $Q \simeq 0.6\,GeV$ and maximal value $\alpha_s^{max} \simeq 0.55$ agrees to calculations in work M. Baldicchi, A.V. Nesterenko, G.M. Prosperi et al., arXiv, 0705.1695 [hep-ph].

Qualitately the result also corresponds to lattice calculations in work E.-M Ilgenfritz, M. Müller-Preussker, A. Sternbeck and A. Schiller, arXiv, hep-lat/0601027.
See also discussion in paper D.V. Shirkov, Eur. Phys. J. C, **22**, 331 (2001).
Note, that we begin plot at Fig.3 starting from $Q = 0.05\,GeV$, because $\alpha_s(u)$ has a pole at very small u, which is analogous to the well-known perturbative pole at $u = 1$. Now this pole is shifted to the far infrared region. One may deal with it using the method proposed in work {D.V. Shirkov and I.L. Solovtsov, Phys. Rev. Lett., **79**, 1209 (1997).} and subtract from (21)

$$\frac{4\,\pi}{b_0\,D\,(u\;-\;u_{00})}\,;\quad u_{00}\;=\;0.005769\,;\quad D\;=\;170.1594\,. \tag{24}$$

This procedure practically does not change the result presented at Fig.3 in the denoted interval of Q. For comparison we present at Fig. 4 the modified $\alpha_s(Q)$ for interval $0.01\,GeV < Q < 1\,GeV$. Value of $\alpha_s(Q)$ at zero reads $\alpha_s(0) = 1.4205$. All values are now expressed in terms of Λ_{QCD}. Emphasize, that there is no additional parameters to describe the non-perturbative infrared region.

$$G = \frac{5.497571}{\Lambda_{QCD}^2}\,. \tag{25}$$

We calculate non-perturbative vacuum average of the third power in gluon field, which is immediately connected with our results. We have

$$< g^3 f_{abc} F^a_{\mu\nu} F^b_{\nu\rho} F^c_{\rho\mu} >= \frac{g^3 G 96 \pi^4}{(2\pi)^8} I_1 I_2 = 110.8\,\Lambda_{QCD}^6;$$

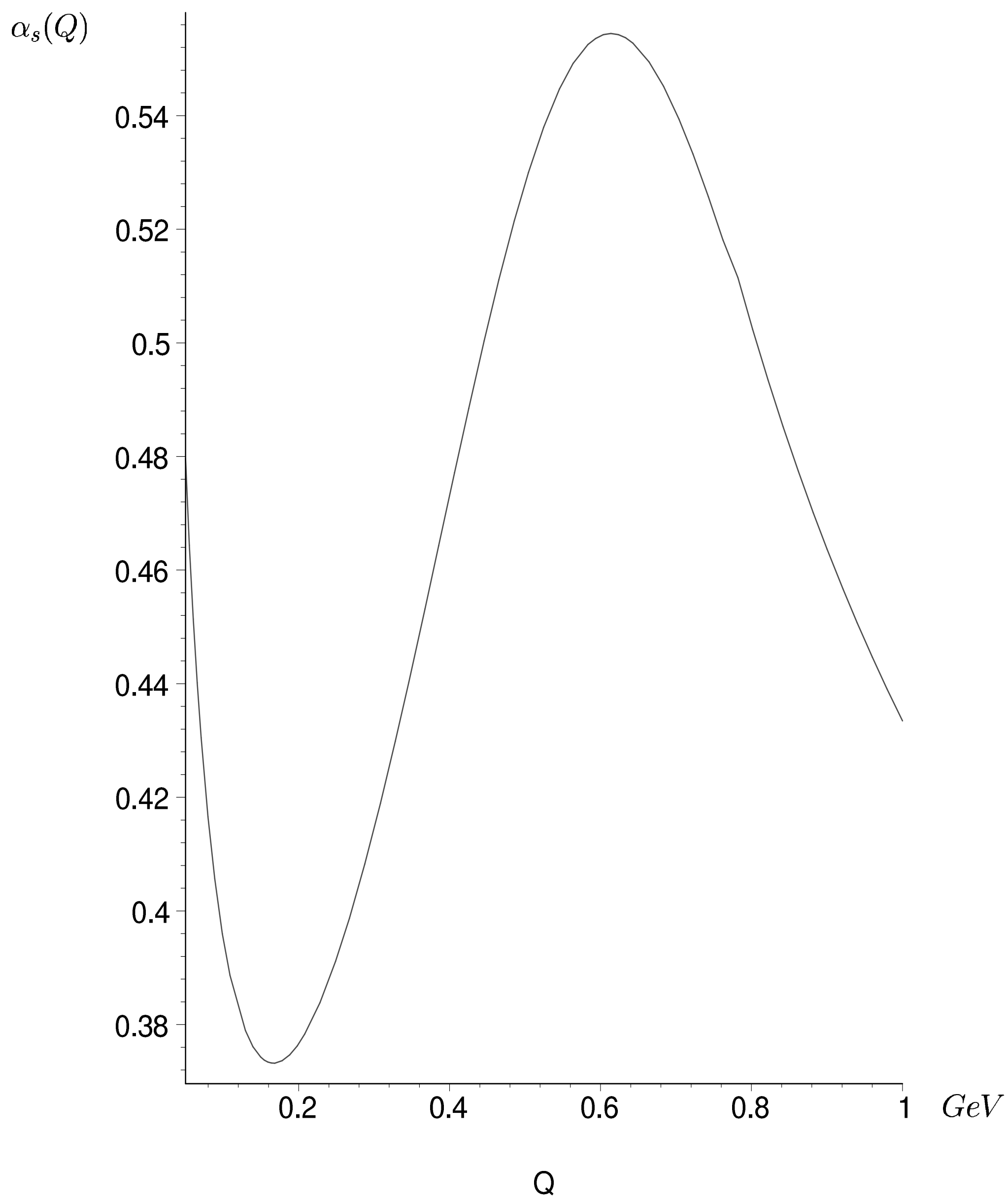
$$
\begin{aligned}
I_1 \;&=\; \int_0^{\frac{4}{3}}(1-\frac{3y}{4})dy - \int_0^1 \frac{(1-y)^2}{(1-y/2)}dy - \qquad (26)\\
&- \int_1^{\frac{4}{3}} \frac{(1-y)^2(1-3y/4)}{y(1-y/2)}dy \;=\; 0.278756\,;\\
I_2 \;&=\; \int_0^Y x^3\,dx\,F(x) \;=\; 0.18062\,\frac{(32\,\pi)^4}{2\,G^4\,N^2}\,.
\end{aligned}
$$

The mean non-perturbative value for α_s turns to be here around 0.5. For this value results for low-energy hadron parameters from works {B.A. A., Phys. Atom. Nucl., **69**, 1588 (2006); B.A.A., M.K. Volkov and I.V. Zaitsev, Int. Journ. Mod. Phys. A, **21**, 5721 (2006).}

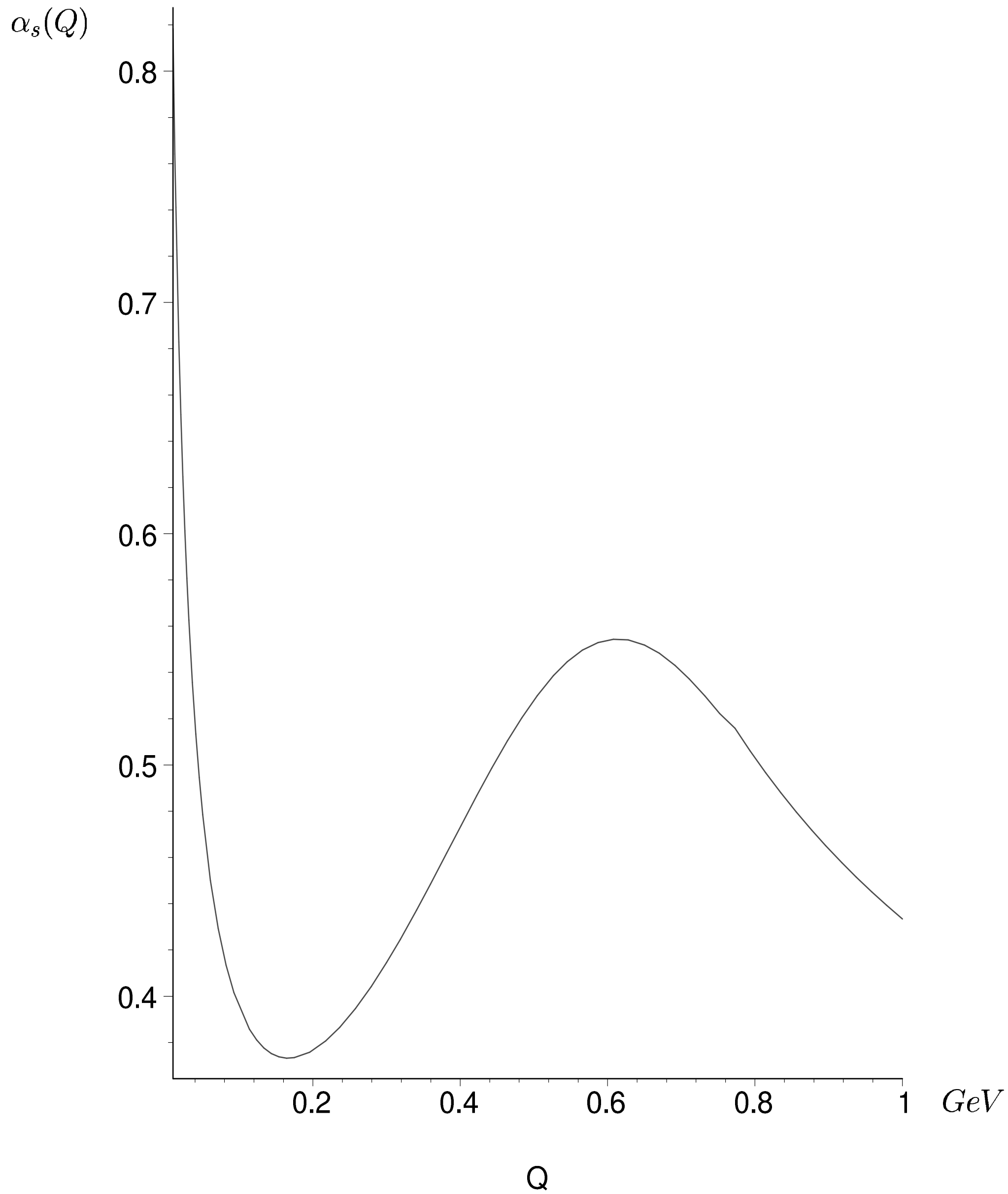
$$
\begin{aligned}
&m_\pi \;=\; 134\,MeV\,;\; f_\pi \;=\; 93\,MeV\,;\\
&m_\sigma \;=\; 460\,MeV\,;\; \Gamma_\sigma \;=\; 580\,MeV\,;\\
&<\bar{q}q> = \,-\,(230\,MeV)^3\,;\; (m_0 \;=\; 19\,MeV)\,;
\end{aligned}
$$

To conclude we would state, that method by Bogolubov being applied to non-perturbative α_s proves its efficiency even in the first approximation, which is considered here. Bearing in mind also results on application of the approach to low-energy hadron physics we would express a hope, that in this way we could obtain the adequate tool to deal non-perturbative effects in QCD and, maybe, in other problems.

{hep-ph/0703237}



$\alpha_s(Q)$
0.54
0.52
0.5
0.48
0.46
0.44
0.42
0.4
0.38
0.2
0.4
0.6
0.8
1
GeV
Q


$\alpha_s(Q)$
0.8
0.7
0.6
0.5
0.4
0.2
0.4
0.6
0.8
1
GeV
Q