

# Light-cone distribution amplitudes for non-relativistic bound states

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- 2 LCDA for a non-relativistic “Pion”
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# Motivation

- Hard exclusive processes (often) factorize:  
$$\text{perturbative scattering amplitudes} \otimes \text{hadronic light-cone distribution amplitudes (LCDAs)}$$
  - Brodsky/Lepage, pQCD, ...
  - QCD factorization in exclusive  $B$  decays (BBNS, SCET ...)
  - Light-cone sum rules.
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- Pion DA fairly well known. (experimental data, lattice, sum rules)
  - LCDAs for **heavy mesons** and **quarkonia** less well understood.

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  - Light-cone sum rules.
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## Our idea:

- Consider non-relativistic limit,  
as initial condition at low scales,  $\mu^2 \sim m^2$ .
- Study radiative QCD corrections:  
(generation of radiative tail from evolution to  $\mu^2 \gg m^2$ ).



- Systematic approximation for LCDAs of heavy quarkonia.\*
- Toy model for LCDAs of light and heavy mesons.
- Playground to test general aspects of QCD factorization.

\* (see also [Ma/Si, hep-ph/0608221], [Braguta et al., hep-ph/0611021, parallel session])

# LCDA for a non-relativistic “Pion”

Formal definition (leading twist):

$(z^2 = 0)$

$$\langle \pi(p) | \bar{q}(z) [z, 0] \not{\gamma}_5 q(0) | 0 \rangle = -i f_\pi(p \cdot z) \int_0^1 du e^{i u p \cdot z} \phi_\pi(u; \mu)$$

$(u = \text{momentum fraction of the quark inside the pion})$

$\leftrightarrow \xi = 1 - 2u$

Non-relativistic limit:

- Bound state of two equal quarks:  $m_\pi \simeq 2m$
- Realistic case:  $\eta_b$  or  $\eta_c$  meson

$$\Rightarrow \phi_\pi(u) \simeq \delta(u - 1/2)$$

Similar considerations for higher-twist LCDAs ...

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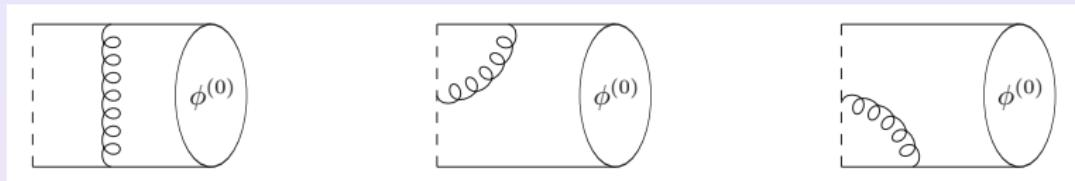
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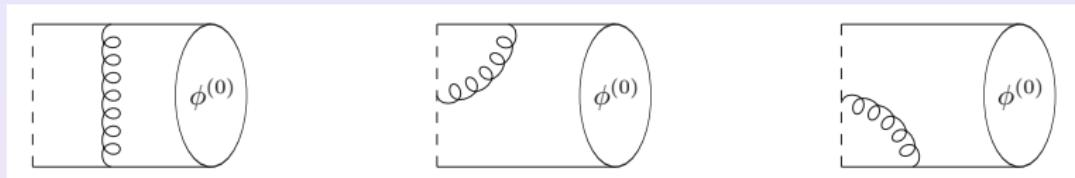
# Radiative corrections from relativistic gluons



- Corrections to the NR limit at low scales:

$$\phi_\pi(u, \mu) = \delta(u - 1/2) + \frac{\alpha_s C_F}{4\pi} \phi_\pi^{(1)}(u, \mu) + \dots$$

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- UV divergences determine evolution to higher scales:

$$\frac{d}{d \ln \mu} \phi_\pi(u; \mu) = \frac{\alpha_s C_F}{\pi} \int_0^1 dv V^{(1)}(u, v) \phi_\pi(v; \mu) + \dots$$

( $V(u, v)$  is the Brodsky-Lepage kernel)

## Result:

- Local limit determines corrections to NR decay constant: ✓

$$f_\pi = f_\pi^{\text{NR}} \left[ 1 - 6 \frac{\alpha_s C_F}{4\pi} + \mathcal{O}(\alpha_s^2) \right]$$

[consistent with earlier result by Braaten/Fleming]

- Remaining (non-local) terms give:

$$\begin{aligned}\phi_\pi^{(1)}(u; \mu) &= \left\{ 4 \ln \left[ \frac{\mu^2}{m^2(1-2u)^2} \right] \left( u + \frac{2u}{1-2u} \right) \theta\left(\frac{1}{2}-u\right) + (u \leftrightarrow \bar{u}) \right\}_{++} \\ &\quad - \left\{ 4 \left( u - \frac{2u}{(1-2u)^2} \right) \theta\left(\frac{1}{2}-u\right) + (u \leftrightarrow \bar{u}) \right\}_{++}\end{aligned}$$

$$\left[ \int_0^1 du \{ \dots \}_{++} f(u) \equiv \int_0^1 du \{ \dots \} (f(u) - f(1/2) - f'(1/2)(u - 1/2)) \right]$$

- Fixed-order result valid for  $\mu_0 = \mathcal{O}(m)$ .
- $\mu$ -dependence consistent with BL evolution. ✓

## Gegenbauer expansion:

- Projection onto Gegenbauer polynomials  $C_n^{(3/2)}(2u - 1)$   
= Eigenfunctions of LO evolution kernel:

$$a_n(\mu) = \frac{2(2n+3)}{3(2+n)(1+n)} \int_0^1 du \phi_\pi(u, \mu) C_n^{(3/2)}(2u - 1)$$

$n =$	0	2	4	6
$a_n^{(0)}$	1	-0.58...	+0.45...	-0.39...
$a_n^{(1)}(m)$	0	+9.16...	-15.37...	+19.83...

⇒ Gegenbauer expansion does not converge well ...

- $a_n$  fall off slower than  $1/n$
- artefact of singular behaviour at  $u = 1/2$  ...

# Alternative Strategy:

- Ansatz:

$$\phi_\pi(u) \equiv \frac{3u\bar{u}}{\Gamma[a, -\ln t_c]} \int_0^{t_c} dt (-\ln t)^{a-1} \left( f(2u-1, it^{1/b}) + f(2u-1, -it^{1/b}) \right)$$

with generating function for Gegenbauer polynomials:

$$f(\xi, \theta) = \frac{1}{(1 - 2\xi\theta + \theta^2)^{3/2}} = \sum_{n=0}^{\infty} C_n^{(3/2)}(\xi) \theta^n$$

- Gegenbauer coefficients given as

$$a_n = \cos\left(\frac{n\pi}{2}\right) \frac{\Gamma[a, -(1+n/b)\ln t_c]}{\Gamma[a, -\ln t_c]} (1+n/b)^{-a}$$

- (Numerically) fix parameter  $a, b, t_c$  from e.g.  $a_2(\mu), a_4(\mu), a_6(\mu)$

⇒ Approximation / Model Parametrization for  $\phi_\pi(u, \mu)$

## Gegenbauer coefficients at large $n$

- For  $t_c \equiv 1$ , recover ansatz in [Ball/Talbot, JHEP 0506 (2005) 063] with power-law fall off

$$|a_n| = \frac{1}{(n/b + 1)^a} \xrightarrow{n \gg b} (n/b)^{-a}$$

- With  $t_c < 1$ , power-law fall-off for  $b \ll n \ll n_{\text{crit}} = -b(1 + 1/\ln t_c)$ , but exponential fall-off for asymptotic values of  $n$ :

$$|a_n| \xrightarrow{n \gg n_{\text{crit}}} -\frac{b(\ln t_c)^{a-1}}{\Gamma[a, -\ln t_c]} \frac{t_c^{n/b}}{n}$$

## Application to $\phi_\pi(u) = \delta(u - 1/2)$

- Fix model parameters from  $a_{2,4,6}$ :

$$a = 0.3962 \dots, \quad b = 0.8045 \dots, \quad t_c = 0.9993 \dots$$

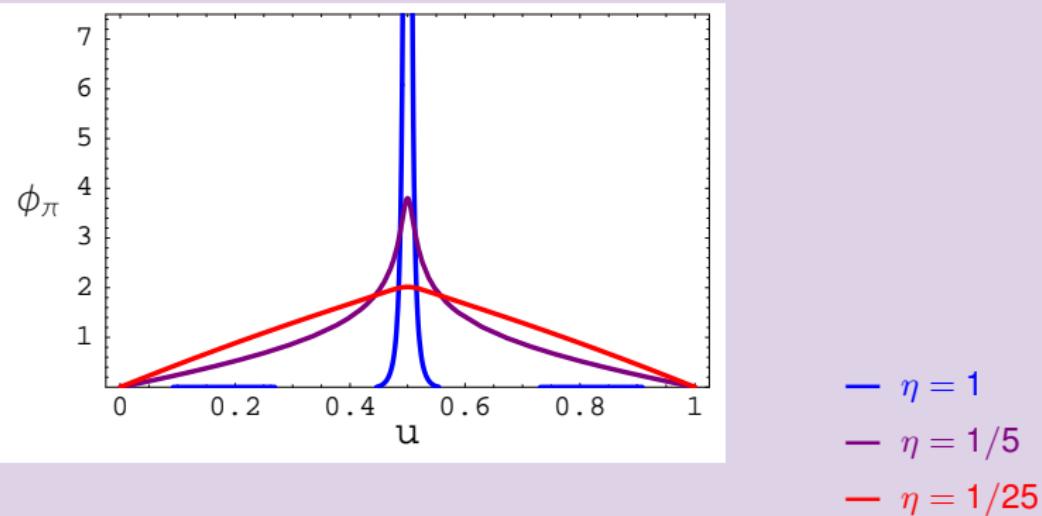
(Critical value  $n_{\text{crit}} = 1149$  and  $a < 1$  reflect bad convergence)

## Evolution with $\eta = \alpha_s(\mu)/\alpha_s(m)$ :

	$a_2$	$a_4$	$a_6$	$a_8$	$a_{10}$	$a_{12}$	$a$	$n_{\text{crit}}$
$\eta = 1$ model	-0.583 *	0.458 *	-0.391 *	0.346 0.346	-0.314 -0.314	0.290 0.289	0.396	1149
$\eta = 1/5$ model	-0.216 *	0.108 *	-0.068 *	0.048 0.048	-0.036 -0.036	0.029 0.028	1.268	45
$\eta = 1/25$ model	-0.080 *	0.025 *	-0.012 *	0.007 0.007	-0.004 -0.004	0.003 0.003	2.145	14

(• parameter  $b$  increases slightly, • parameter  $t_c$  decreases slightly)

# LL Evolution of $\phi_\pi(u, \mu)$ for model parametrization



inverse moment:

$$\int_0^1 du \frac{\phi_\pi(u)}{u} = 2.00 \text{ (2.55, 2.81)}$$

Similar results for  $m_1 \neq m_2$ :

- Toy model for Kaon distribution amplitude.
- Dependence of  $a_1$  on quark masses and  $\alpha_s$ .
- Constraints from equations of motion ...

# LCDA for a non-relativistic “ $B$ -Meson”

Formal definition (in HQET):

[Grozin/Neubert 96, Beneke/TF 00]

$$\begin{aligned} & \langle 0 | (\bar{q})_\beta(z) [z, 0] (h_v)_\alpha(0) | B(v) \rangle \\ = & -\frac{i\hat{f}_B(\mu)M_B}{4} \int_0^\infty d\omega e^{-i\omega t} \left[ \frac{1+\gamma}{2} \left\{ 2\phi_B^+(\omega) + \frac{\phi_B^-(\omega) - \phi_B^+(\omega)}{t} \not{z} \right\} \gamma_5 \right]_{\alpha\beta} \end{aligned}$$

- $t = v \cdot z$ , with  $v^\mu$  = heavy-quark velocity.
- $\omega$  : light-cone momentum of spectator quark.

Non-relativistic limit:

- Bound state of one heavy and one light massive quark:

$$M_B \simeq (M + m), \quad m \ll M. \quad (\text{Realistic case: } B_c \text{ mesons})$$

$$\Rightarrow \phi_B^+(\omega) \simeq \phi_B^-(\omega) \simeq \delta(\omega - m)$$

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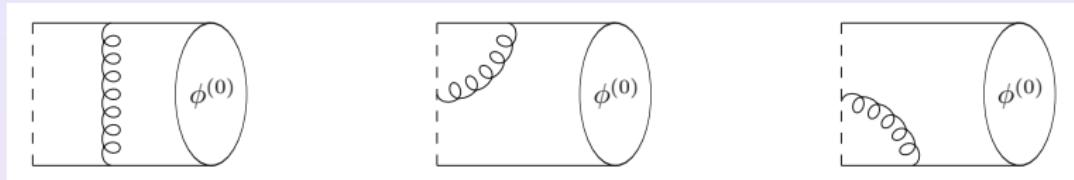
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# Radiative corrections from relativistic gluons



## Result:

- (scale-dependent) HQET decay constant:

✓

$$\hat{f}_M(\mu) = f_M^{\text{NR}} \left[ 1 + \frac{\alpha_s C_F}{4\pi} \left( 3 \ln \frac{\mu}{m} - 4 \right) + \mathcal{O}(\alpha_s^2) \right]$$

- Corrections to  $\phi_B^+(\omega)$ : (contain  $\ln^2 \mu$ ) ✓

$$\begin{aligned} \frac{\phi_B^{(+,1)}(\omega; \mu)}{\omega} &= 2 \left[ \left( \ln \left[ \frac{\mu^2}{(\omega-m)^2} \right] - 1 \right) \left( \frac{\theta(m-\omega)}{m(m-\omega)} + \frac{\theta(\omega-m)}{\omega(\omega-m)} \right) \right]_+ + 4 \left[ \frac{\theta(2m-\omega)}{(\omega-m)^2} \right]_{++} \\ &\quad + \frac{4 \theta(\omega-2m)}{(\omega-m)^2} - \frac{\delta(\omega-m)}{m} \left( \frac{1}{2} \ln^2 \left[ \frac{\mu^2}{m^2} \right] - \ln \left[ \frac{\mu^2}{m^2} \right] + \frac{3\pi^2}{4} + 2 \right) \end{aligned}$$

## Inverse moments of $\phi_B^+(\omega)$ :

- Relevant phenomenological parameters in factorization theorems for exclusive  $B$ -decays:

$$\frac{1}{\lambda_B} \equiv \int_0^\infty d\omega \frac{\phi_B^+(\omega; \mu)}{\omega} = \frac{1}{m} \left( 1 - \frac{\alpha_s C_F}{4\pi} \left[ \frac{1}{2} \ln^2 \frac{\mu^2}{m^2} - \ln \frac{\mu^2}{m^2} + \frac{3\pi^2}{4} - 2 \right] \right),$$

$$\sigma_B \equiv \lambda_B \int_0^\infty d\omega \frac{\phi_B^+(\omega; \mu)}{\omega} \ln \frac{\mu}{\omega} = \ln \frac{\mu}{m} + \frac{\alpha_s C_F}{4\pi} [8\zeta(3)].$$

## Model-independent properties of $\phi_B^+(\omega)$

- Scale-dependence described by Lange-Neubert evolution: ✓

$$\frac{d}{d \ln \mu} \phi_B^+(\omega; \mu) = -\frac{\alpha_s C_F}{4\pi} \int_0^\infty d\omega' \gamma_+^{(1)}(\omega, \omega'; \mu) \phi_B^+(\omega'; \mu)$$

- Positive moments of  $\phi_B^+(\omega)$  depend on UV cut-off ✓

$$\int_0^{\Lambda_{\text{UV}}} d\omega \phi_B^+(\omega; \mu) \simeq 1 - \frac{\alpha_s C_F}{4\pi} \left[ \frac{1}{2} \ln^2 \frac{\mu^2}{\Lambda_{\text{UV}}^2} + \ln \frac{\mu^2}{\Lambda_{\text{UV}}^2} + \frac{\pi^2}{12} \right],$$

$$\frac{1}{\Lambda_{\text{UV}}} \int_0^{\Lambda_{\text{UV}}} d\omega \omega \phi_B^+(\omega; \mu) \simeq \frac{\alpha_s C_F}{4\pi} \left[ 2 \ln \frac{\mu^2}{\Lambda_{\text{UV}}^2} + 6 \right] + \mathcal{O}(\alpha_s^2)$$

[Lee/Neubert 2005]

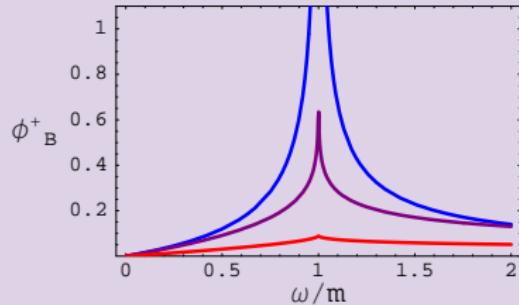
# LL Evolution of $\phi_B^+(\omega, \mu)$

Closed form for solution of RG equation:

[Lee/Neubert 05]

$$\phi_B^+(\omega, \mu) = e^{V_+ - 2\gamma_E g} \frac{\Gamma(2-g)}{\Gamma(g)} \int_0^\infty \frac{d\omega'}{\omega'} \phi_B^+(\omega', \mu_0) \left(\frac{\omega_>}{\mu_0}\right)^g \frac{\omega_<}{\omega_>} {}_2F_1\left(1-g, 2-g; 2; \frac{\omega_<}{\omega_>}\right)$$

with  $\omega_< = \min(\omega, \omega')$ ,  $\omega_> = \max(\omega, \omega')$

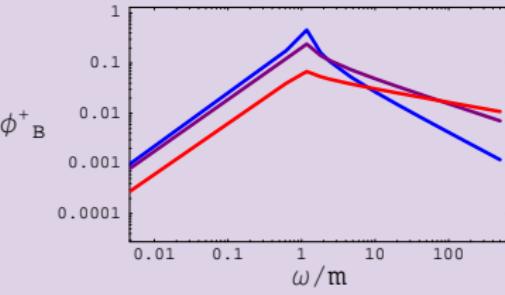


—  $\eta = 1/2$

—  $\eta = 1/5$

—  $\eta = 1/10$

( $\alpha_s(\mu_0) := 1$ )



## Similar results for $\phi_B^-(\omega, \mu)$ :

- $\mathcal{O}(\alpha_s)$  corrections at  $\mu = m$ .
- Corrections to Wandzura-Wilczek relation: [Beneke/TF 2000]

$$\int_0^\omega d\eta (\phi_B^-(\eta) - \phi_B^+(\eta)) \simeq \omega \phi_B^-(\omega) - m \phi_B^+(\omega)$$

- Evolution for  $\phi_B^-(\omega, \mu)$  (neglecting 3-particle DAs)

# Summary

- LCDAs provide important phenomenological input to factorization theorems for hard exclusive reactions.
- NR expansion can be employed for bound states of heavy quarks.
- Radiative corrections from relativistic gluons can be included.

[There are also  $\mathcal{O}(|\vec{V}_{\text{NR}}|)$  corrections from the NR expansion. → Braguta]

- Finite order  $\alpha_s$  corrections at  $\mu_0 = m_{\text{red}}$ .
- Evolution towards higher factorization scales  $\mu \sim Q_{\text{hard}}$ .

- (Quantitative) predictions for heavy quarkonia ( $b\bar{b}$ ,  $c\bar{c}$ ,  $b\bar{c}$ ).
- Toy-model for light mesons and  $B_q$ -mesons.
- Cross-check of model-independent properties.