Finite-Size Scaling from the Non-Perturbative Renormalization Group

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Introduction

A powerful method in the research of phase transitions in lattice QCD is a finite-size scaling analysis. From the scaling behavior, critical exponents can be tested and the order and the universality class of a phase transition can be established [1]. To do this, the critical exponents have to be known accurately, and knowledge of the universal finite-size scaling functions is advantageous.

For phase transitions in the O(N) universality classes, finite size scaling has been investigated mainly by lattice simulations of $O(N)$-spin models [2]. While very successful, these results are usually limited to the critical temperature or the temperature at which the susceptibilities diverge. Thus lattice data away from these points cannot be mapped onto the resulting universal scaling functions, and a large number of data points is necessary to establish the scaling behavior.

We have previously used non-perturbative renormalization group methods to investigate finite-size effects on particle masses and on the chiral phase transition [3], and we have now established finite-size scaling behavior for an $O(N)$-model in a non-perturbative renormalization group calculation. The result is also a consistency check on our values for the critical exponents.

We calculate finite-size effects on the critical scaling behavior and the resulting universal scaling functions for the $O(4)$ universality class for a very wide range of temperatures and values of the symmetry-breaking parameter (in QCD, this is the quark mass). Our results are suitable for a comparison to lattice susceptibilities to check consistency of the finite-size scaling behavior with $O(N)$ scaling.

Determination of Critical Exponents

We determine the critical exponents (β, ν, δ) for an $O(4)$-model by (a) evaluating the logarithmic derivative of the order parameter or the susceptibility close to the critical temperature (β, ν), or field $H = 0$ (δ), (b) fitting to the order parameter or the susceptibility as a function of temperature (β, ν) or field H (δ). For comparison, we give values of the critical exponents for the O(4) model from other methods.

		ν	β	η	δ
J. Engels *et al.* [2]	lattice	0.7423	0.380	(0.024)	4.86
K. Kanaya, and S. Kaya (1995)	lattice	0.7479(90)	0.3836(46)	(0.0254(38))	4.851(22)
H. G. Ballesteros *et al.* [4]	lattice	0.7525(10)	(0.3907(10))	0.0384(12)	**(4.778(8))**
N. Tetradis, and C. Wetterich (1994)	RG	0.791	0.409	0.034	(4.80)
D. F. Litim, and J. M. Pawlowski [5]	RG, LPA	0.8043	(0.4022)	–	(5.00)
our work, (a) from derivative	RG, LPA	0.8053(6)	0.4030(3)	(0.0046(4))	4.9727(5)
our work, (b) from fit for $H = 0$	RG, LPA	0.7953(6)	0.4051(2)	–	–
our work, (b) from fit for $H = 1.0 \cdot 10^{-9}$	RG, LPA	0.7920(13)	0.4108(4)	(0.027(1))	4.8409(15)

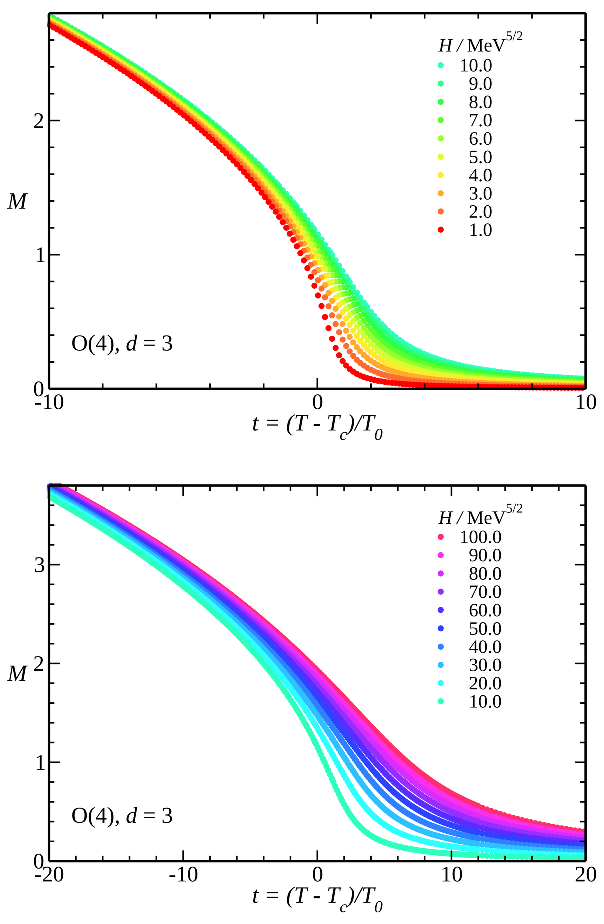
Results for ν obtained from a fit are consistently smaller than those obtained from the derivative close to the critical point. We use an approximation (LPA) in which $\eta = 0$, which likely accounts for much of the difference to the lattice results. In the following, we use the results from (a) for the evaluation.

Scaling in Infinite Volume $V \to \infty$

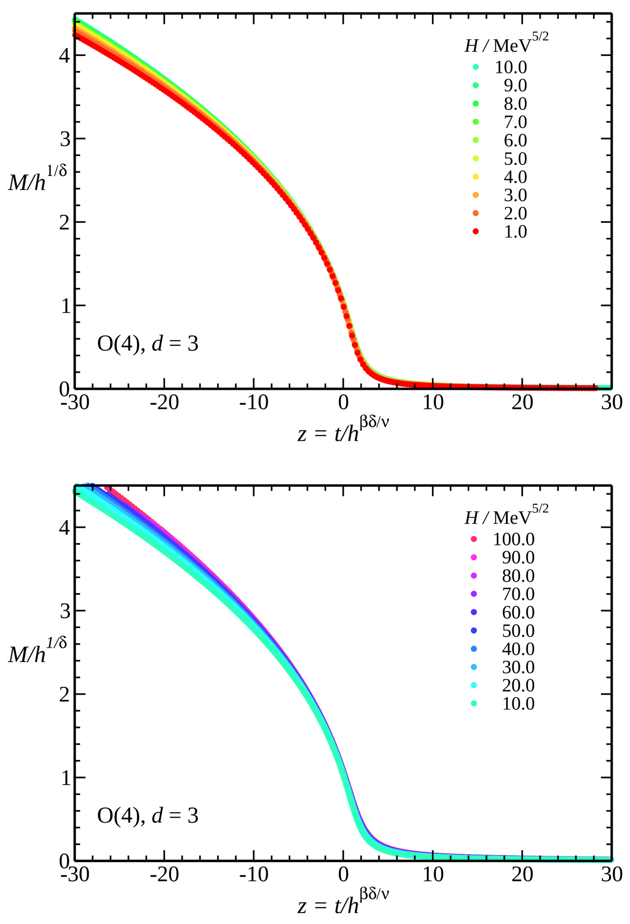
- Critical points (e.g. continuos phase transitions) are characterized by a diverging correlation length ξ
- Critical long-range fluctuations lead to universal behavior: *universality classes* of systems
- transitions characterized by only a few *critical exponents* and *scaling functions* $f(z)$

$$t = (T - T_c)/T_0, \qquad h = H/H_0, \qquad z = \frac{t}{h^{1/(\beta\delta)}}, \qquad \xi(t) \sim t^{-\nu}$$

$$M(t,h) = h^{1/\delta} f(z), \qquad M(t, h=0) = (-t)^{\beta}, \qquad \chi = h^{1/\delta - 1} f_\chi(z)$$



Order parameter M vs. temperature $t = (T - T_c)/T_0$ for different values of the symmetry-breaking field H.



Scaled order parameter $M/h^{1/\delta}$ vs. $z = t/h^{\beta\delta/\nu}$ different values of the symmetry-breaking field H.

Finite-Size Scaling

- Universal behavior requires divergence of the correlation length ξ
- Finite volume size L cuts off the critical fluctuations
- Universal scaling behavior is therefore affected if correlation length $\xi \sim L$, depends on ratio ξ/L

Finite-Size Scaling hypothesis (Fisher): The ratio of thermodynamic quantities $(M, \chi, \ldots)$ in the finite-size system and the infinite-size system is a function of only the ratio ξ/L:

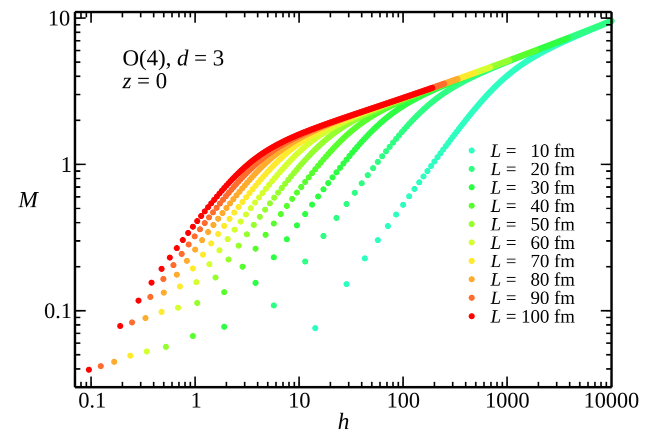
$$\frac{M_L(t)}{M_\infty(t)} = \mathcal{F}\left(\frac{L}{\xi(t)}\right), \qquad \xi(t) \sim t^{-\nu}, \qquad M(t,h) = h^{1/\delta} f(z), \qquad z = \frac{t}{h^{1/(\beta\delta)}}$$

Idea for obtaining the universal Finite-Size Scaling functions:

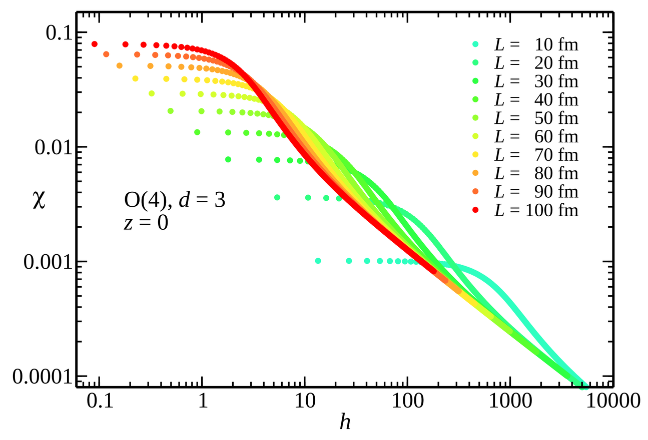
- keep $L/\xi = \text{const.}$ $\to$ vary $t \sim L^{1/\nu}$
- keep $z = t/h^{1/(\beta\delta)} = \text{const.}$ $\to$ vary $h \sim L^{-\beta\delta/\nu}$

Finite-Size Scaling Functions depend only on $hL^{\beta\delta/\nu}$ (for any given value of z):

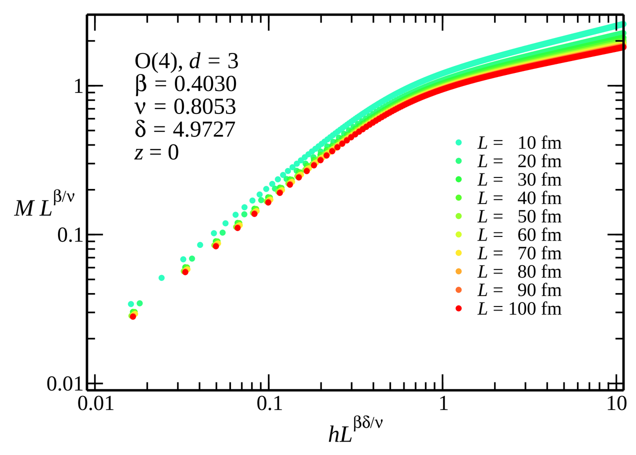
$$L^{\beta/\nu} M = Q_M^{(0)}(z, hL^{\beta\delta/\nu}) + \frac{1}{L^\omega} Q_M^{(1)}(z, hL^{\beta\delta/\nu}) + \ldots$$
$$L^{\gamma/\nu} \chi = Q_\chi^{(0)}(z, hL^{\beta\delta/\nu}) + \frac{1}{L^\omega} Q_\chi^{(1)}(z, hL^{\beta\delta/\nu}) + \ldots$$



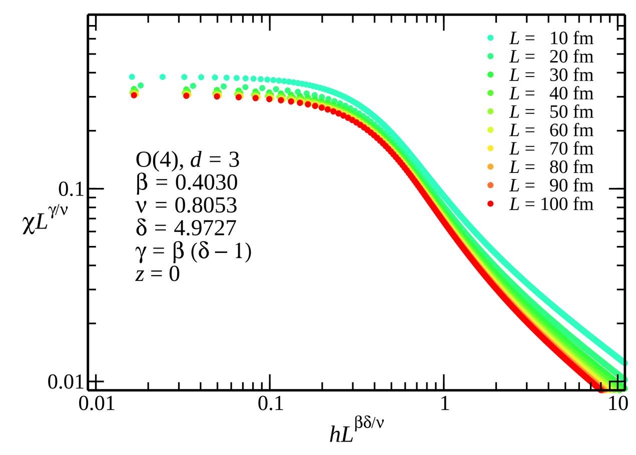
Order parameter M vs. H at the critical temperature ($z = 0$) for $L = 10 - 100$ fm.



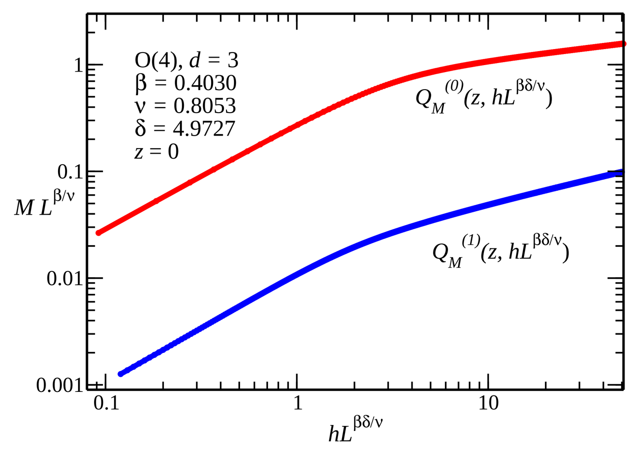
Susceptibility χ vs. H at the critical temperature ($z = 0$) for $L = 10 - 100$ fm.



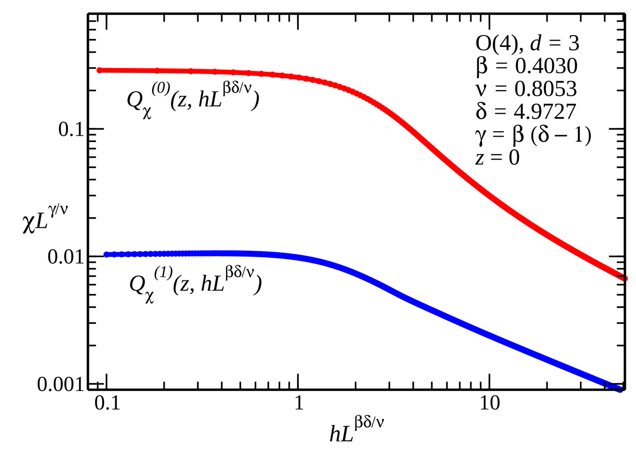
Finite-size scaled order parameter $ML^{\beta/\nu}$ vs. $hL^{\beta\delta/\nu}$ for $L = 10 - 100$ fm.



Finite-size scaled susceptibility $\chi L^{\gamma/\nu}$ vs. $hL^{\beta\delta/\nu}$ for $L = 10 - 100$ fm.



Scaling functions $Q_M^{(0)}(z, hL^{\beta\delta/\nu})$ and $Q_M^{(1)}(z, hL^{\beta\delta/\nu})$ vs. $hL^{\beta\delta/\nu}$ at $z = 0$.



Scaling functions $Q_\chi^{(0)}(z, hL^{\beta\delta/\nu})$ and $Q_\chi^{(1)}(z, hL^{\beta\delta/\nu})$ vs. $hL^{\beta\delta/\nu}$ at $z = 0$.

The finite-size scaled results (middle panels) do not fit perfectly due to sub-leading scaling corrections. Sub-leading critical scaling behavior is characterized by the first irrelevant critical exponent ω. We determine from a fit $\omega = 0.744(30)$ from M and $\omega = 0.738(36)$ from χ. This is consistent with the value from the RG fixed point in this RG scheme $\omega = 0.7338$ [5]. We use this value to determine the scaling functions.

Above only an example for the scaling functions at the critical temperature $T = T_c$ $(z = 0)$ is given, but we have obtained results over a wide range of z-values (away from the critical temperature).

Using these results, lattice data away from the critical coupling can be included in a comprehensive finite-size scaling analysis away from a single curve of fixed $z = z_0$.

Conclusions

We have calculated the critical exponents and the universal scaling functions for the $O(4)$ universality class in $d = 3$ dimensions with a non-perturbative Renormalization Group scheme. We demonstrate scaling behavior in infinite volume and thus validate our results for the critical exponents. The appearance of universal scaling behavior shows that our RG scheme takes long-range fluctuations into account properly.

We have further demonstrated finite-size scaling in this model by implementing our RG scheme in a finite volume. We have calculated the universal finite-size scaling functions for the order parameter M and the susceptibility χ for a wide range of z-values (although here we show only the results for $z = 0$).

These results can be compared to results from lattice QCD to check the compatibility of the scaling behavior with the $O(4)$ universality class and to classify the phase transition. The comparison with known scaling functions can establish the scaling behavior with fewer data points.

We will also present the corresponding results for the $O(2)$- and $O(3)$-universality classes. We plan to improve our results by going beyond the approximation $\eta = 0$. This requires to take the momentum dependence of the critical fluctuations into account; work on this is completed for infinite volume.

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