STABILITY OF SALPETER SOLUTIONS

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Motivation

In the framework of the instantaneous approximation to the Bethe–Salpeter formalism for the description of bound states within quantum field theories, depending on the Lorentz structure of the Bethe–Salpeter interaction kernel the solutions of the (full) Salpeter equation with some confining interactions may exhibit instabilities [1], possibly related to the Klein paradox, signalling the decay of some states assumed to be bound by the confining interactions. Such instabilities are found in numerical studies [1] of the Salpeter equation.

The perhaps simplest scenario allowing for the analytic investigation of this problem is set by the reduced Salpeter equation [2] with harmonic-oscillator interaction. In this case Salpeter's integral equation becomes a second-order homogeneous linear differential equation, accessible to standard techniques. There one can hope to be able to decide unambiguously whether this setting poses a well-defined (eigenvalue) problem the solutions of which yield stable bound states corresponding to real energy eigenvalues bounded from below.

Reduced Salpeter Equation for Interaction Kernels of Pure Harmonic-Oscillator Type

Assuming, as usual, the Lorentz structures of the effective couplings of both fermion and antifermion to be represented by identical Dirac matrices Γ and denoting the associated Lorentz-scalar interaction function by $V_\Gamma(\boldsymbol{p},\boldsymbol{q})$, the reduced Salpeter equation [2] describing bound states composed of fermion and corresponding antifermion (of mass m and relative momentum $\boldsymbol{p}$) reads for a bound state with mass eigenvalue M in its center-of-momentum frame

$$(M-2\,E)\,\Phi(\boldsymbol{p})=\Lambda^+(\boldsymbol{p})\,\gamma_0\int\frac{\mathrm{d}^3q}{(2\pi)^3}\sum_\Gamma V_\Gamma(\boldsymbol{p},\boldsymbol{q})\,\Gamma\,\Phi(\boldsymbol{q})\,\Gamma\,\Lambda^-(\boldsymbol{p})\,\gamma_0\ ,$$

with one-particle kinetic energies E and energy projectors $\Lambda^\pm(\boldsymbol{p})$ defined by

$$E\equiv\sqrt{p^2+m^2}\ ,\quad p\equiv|\boldsymbol{p}|\ ,\quad\text{and}\quad\Lambda^\pm(\boldsymbol{p})\equiv\frac{E\pm\gamma_0\,(\boldsymbol{\gamma}\cdot\boldsymbol{p}+m)}{2\,E}\ .$$

Let the Bethe–Salpeter kernel be of convolution type, $V_\Gamma(\boldsymbol{p},\boldsymbol{q})=V_\Gamma(\boldsymbol{p}-\boldsymbol{q})$, arising from a central potential $V(r)$, $r\equiv|\boldsymbol{x}|$, in configuration space. Then, for a harmonic-oscillator potential $V(r)=a\,r^2$, $a\neq 0$, the reduced Salpeter equation becomes a second-order differential equation utilizing the operator

$$D\equiv\frac{\mathrm{d}^2}{\mathrm{d}p^2}+\frac{2}{p}\,\frac{\mathrm{d}}{\mathrm{d}p}\ .$$

In order to make contact with related previous analyses [3–6], we present our line of argument for fermion–antifermion bound states of total spin J, parity $P=(-1)^{J+1}$, and charge-conjugation quantum number $C=(-1)^J$, called ${}^1J_J$ spectroscopically. Due to the projectors $\Lambda^\pm(\boldsymbol{p})$, the Salpeter amplitudes $\Phi(\boldsymbol{p})$ describing these states contain only one independent component $\phi(\boldsymbol{p})$:

$$\Phi(\boldsymbol{p})=2\,\phi(\boldsymbol{p})\,\Lambda^+(\boldsymbol{p})\,\gamma_5\ .$$

More specifically, we consider pseudoscalar (${}^1\mathrm{S}_0$) bound states: $J^{PC}=0^{-+}$. Stripping off its angular variables [7] turns such harmonic-oscillator reduced Salpeter equation into the eigenvalue equation of a Schrödinger operator $\mathcal{H}$:

$$\mathcal{H}\,\phi(p)=M\,\phi(p)\ .$$

It is a straightforward task to work out all the Hamiltonians $\mathcal{H}$ associated to the most popular choices of the Lorentz structure of Bethe–Salpeter kernels:

$\Gamma\otimes\Gamma$		$\mathcal{H}$
$1\otimes 1$	(Lorentz scalar)	$2\,E+a\left(\dfrac{2\,p^2+3\,m^2}{2\,E^4}+\dfrac{m^2}{E}\,D\dfrac{1}{E}\right)$
$\gamma^0\otimes\gamma^0$	(time-component Lorentz vector)	$2\,E+a\left(\dfrac{2\,p^2+3\,m^2}{2\,E^4}-D\right)$
$\gamma_\mu\otimes\gamma^\mu$	(Lorentz vector)	$2\,E+a\left(\dfrac{m^2}{E}\,D\dfrac{1}{E}-2\,D\right)$
$\gamma_5\otimes\gamma_5$	(Lorentz pseudoscalar)	$2\,E+a\,\dfrac{2\,p^2+3\,m^2}{2\,E^4}$
$\frac{1}{2}\,(\gamma_\mu\otimes\gamma^\mu+\gamma_5\otimes\gamma_5-1\otimes 1)$	[8]	$2\,E-a\,D$

Spectral Properties

For the various Dirac structures of the Bethe–Salpeter kernel, the spectra of the differential operators $\mathcal{H}$ exhibit the following stability-relevant features:

- All Hamiltonians $\mathcal{H}$ are self-adjoint, as the differential operators D and $m^2\,E^{-1}\,D\,E^{-1}$ as well as the multiplication by any real-valued function define self-adjoint operators. Hence, the corresponding spectra are real.

  For reasonable kernels, involving potential functions $V_\Gamma(\boldsymbol{p},\boldsymbol{q})$ satisfying $V_\Gamma^*(\boldsymbol{q},\boldsymbol{p})=V_\Gamma(\boldsymbol{p},\boldsymbol{q})$ and coupling matrices Γ satisfying $\gamma_0\,\Gamma^\dagger\,\gamma_0=\pm\Gamma$, the reality of all eigenvalues M follows also from a relation [7] obeyed by any Salpeter amplitude $\Phi(\boldsymbol{p})$ that solves the reduced Salpeter equation:

$$M\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,\mathrm{Tr}\left[\Phi^\dagger(\boldsymbol{p})\,\Phi(\boldsymbol{p})\right]=2\int\frac{\mathrm{d}^3p}{(2\pi)^3}\,E\,\mathrm{Tr}\left[\Phi^\dagger(\boldsymbol{p})\,\Phi(\boldsymbol{p})\right]$$
$$+\int\frac{\mathrm{d}^3p}{(2\pi)^3}\int\frac{\mathrm{d}^3q}{(2\pi)^3}\sum_\Gamma V_\Gamma(\boldsymbol{p},\boldsymbol{q})\,\mathrm{Tr}\left[\Phi^\dagger(\boldsymbol{p})\,\gamma_0\,\Gamma\,\Phi(\boldsymbol{q})\,\Gamma\,\gamma_0\right].$$

- Both for the Lorentz pseudoscalar $\Gamma\otimes\Gamma = \gamma_5\otimes\gamma_5$ and, if $m = 0$, for the Lorentz scalar $\Gamma\otimes\Gamma = 1\otimes 1$ the Hamiltonians $\mathcal{H}$ are pure multiplication operators, with purely continuous spectrum. Bound states do not exist.

- For the time-component Lorentz vector $\Gamma\otimes\Gamma = \gamma^0\otimes\gamma^0$, for the (in fact, simple) Lorentz structure $\Gamma\otimes\Gamma = \frac{1}{2}\,(\gamma_\mu\otimes\gamma^\mu+\gamma_5\otimes\gamma_5-1\otimes 1)$ [8], and, if $m = 0$, for the Lorentz vector $\Gamma\otimes\Gamma = \gamma_\mu\otimes\gamma^\mu$ the Hamiltonians $\mathcal{H}$ form ($\ell = 0$) Schrödinger operators with a positive, infinitely rising potential $V(p) \to \infty$ for $p \to \infty$, provided, of course, the signs of the couplings a are chosen appropriately. These operators have entirely discrete spectra bounded from below; all the bound states may be expected to be stable.

- For $m \neq 0$, because of the presence of the operators $m^2\,E^{-1}\,D\,E^{-1}$ the Hamiltonians $\mathcal{H}$ corresponding to both Lorentz scalar $\Gamma\otimes\Gamma = 1\otimes 1$ and Lorentz vector $\Gamma\otimes\Gamma = \gamma_\mu\otimes\gamma^\mu$ are not standard-Schrödinger operators. In these cases, however, by suitable redefinition of the radial amplitudes $\phi(p)$, the radial differential equations may be transformed to eigenvalue equations of ($\ell = 0$) Schrödinger operators $\mathcal{K} \equiv -D+U(p;M)$ making use of effective potentials $U(p;M)$ involving the mass M as parameter. [As may be guessed from the form of the corresponding Hamiltonian $\mathcal{H}$, for the Lorentz scalar the transformation simply reads $\phi(p) \to E\,\phi(p)$.]

  For given M and appropriate sign of a, the effective potentials $U(p;M)$ are bounded from below and behave like $U(p) \to \infty$ for $p \to \infty$. Thus, the spectra of both auxiliary Hamiltonians $\mathcal{K}$ consist entirely of discrete M-dependent eigenvalues. The derivatives of all latter eigenvalues with respect to M are strictly definite for all M. The bound-state masses M, defined by the zeroes of the eigenvalues of $\mathcal{K}$, must then be also discrete. Since all eigenvalues of $\mathcal{K}$ are strictly decreasing functions of M, a closer inspection proves all bound-state masses M to be bounded from below.

In summary, given the semiboundedness of all our Hamiltonians $\mathcal{H}$ entering in the radial equations the "harmonic-oscillator reduced Salpeter equation" poses (at least for a wide class of Lorentz structures) a well-defined problem, with solutions giving stable bound states related to a real discrete spectrum.

Generalization to (Full) Salpeter Equation

Clearly, a similar discussion may be envisaged for the full Salpeter equation; there, however, these spectral analyses will be somewhat more complicated:

- Although the squares of the mass eigenvalues, M^2, are guaranteed to be real [9], the spectrum is in general not necessarily real and, even in those cases where it may be shown to be real, it is not bounded from below [9]. In particular, for the maybe most important example of Bethe–Salpeter kernels involving only coupling matrices Γ satisfying $\gamma_0\,\Gamma^\dagger\,\gamma_0 = \pm\Gamma$ and potential functions $V_\Gamma(\boldsymbol{p},\boldsymbol{q})$ satisfying $V_\Gamma^*(\boldsymbol{p},\boldsymbol{q}) = V_\Gamma(\boldsymbol{p},\boldsymbol{q}) = V_\Gamma(\boldsymbol{q},\boldsymbol{p})$ the spectrum of mass eigenvalues M consists (in the complex-M plane) of real opposite-sign pairs $(M, -M)$ and imaginary points $M = -M^*$.
- Full-Salpeter amplitudes have more than one independent components. Thus, any full Salpeter equation entails a set of second-order differential equations or—equivalently—a single higher-order differential equation.

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