

The QCD Potential

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Summary

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1.1 Effective Field Theories

1.2 pNRQCD

2. Potential: calculation in PT

2.1 Static potential

3. Applications

4. Potential in the non-perturbative regime

4.1 Lattice results 2006

5. Conclusions

Bibliography

- (1) N. Brambilla, X. Garcia i Tormo, J. Soto and A. Vairo
The logarithmic contribution to the QCD static energy at N^4LO
Phys. Lett. B 647 (2007) 185 [arXiv:hep-ph/0610143](#).
- (2) N. Brambilla, A. Pineda, J. Soto and A. Vairo
Potential NRQCD: an effective theory for heavy quarkonium
Nucl. Phys. B 566 (2000) 275 [arXiv:hep-ph/9907240](#).
- (3) N. Brambilla, A. Pineda, J. Soto and A. Vairo
The infrared behaviour of the static potential in perturbative QCD
Phys. Rev. D 60 (1999) 091502 [arXiv:hep-ph/9903355](#).
- (4) N. Brambilla, A. Pineda, J. Soto and A. Vairo
Effective field theories for heavy quarkonium
Reviews of Modern Physics 77 (2005) 1423 [arXiv:hep-ph/0410047](#).
- (5) N. Brambilla, M. Krämer, R. Mussa, A. Vairo *et al.*
Heavy Quarkonium Physics
CERN Yellow Report, CERN-2005-005, (CERN, Geneva, 2005) 487 p.
[arXiv:hep-ph/0412158](#).

1. Definition

The potential is what to write in a Schrödinger equation

$$E \phi = \left(\frac{p^2}{m} + V(r) \right) \phi$$

In a full theory, V must come from a double expansion:

- a non-relativistic expansion $\sim p/m, rm$: $V \rightarrow V^{(0)} + V^{(1)}/m + \dots$;
- an expansion in $E r$, since V is a function of r (or p at h.o. in the non-relativistic expansion): $V \rightarrow V +$ energy-dependent effects (e.g. Lamb-shift).

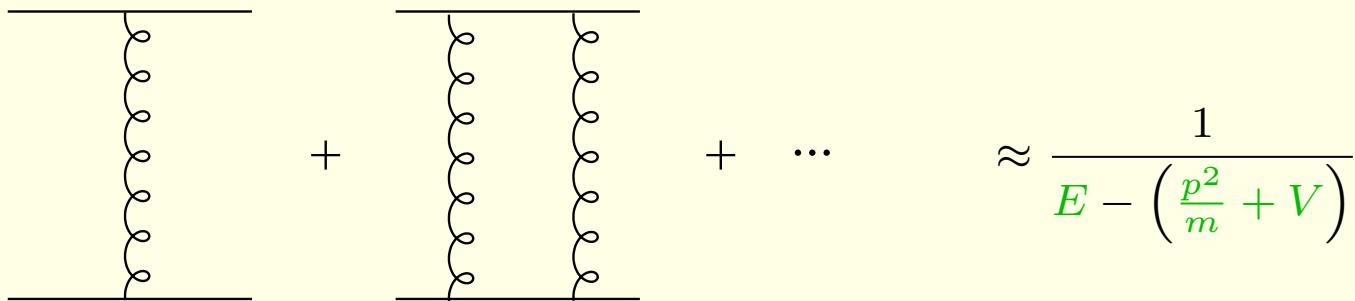
A potential V describes the interaction of a non-relativistic bound state, $p \sim mv$, $E \sim mv^2$, $v \ll 1$, once the expansions in mv/m and mv^2/mv have been exploited.

Non-relativistic scales in QCD

Near threshold:

$$E \approx 2m + \frac{p^2}{m} + \dots \quad \text{with} \quad v = \frac{p}{m} \ll 1$$

- The perturbative expansion breaks down when $\alpha_s \sim v$:

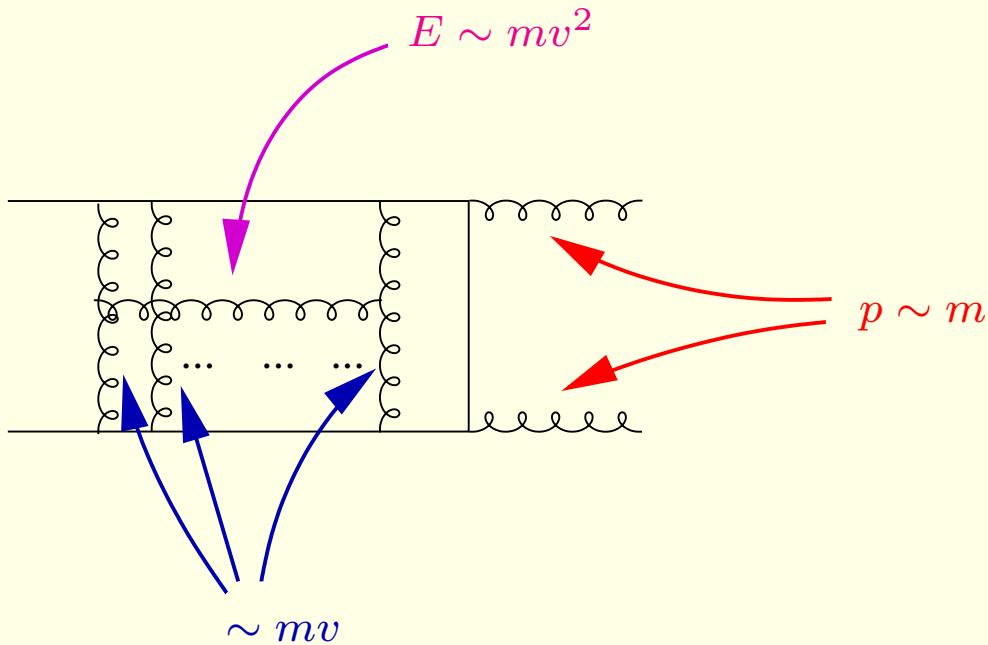


$$\alpha_s \left(1 + \frac{\alpha_s}{v} + \dots \right)$$

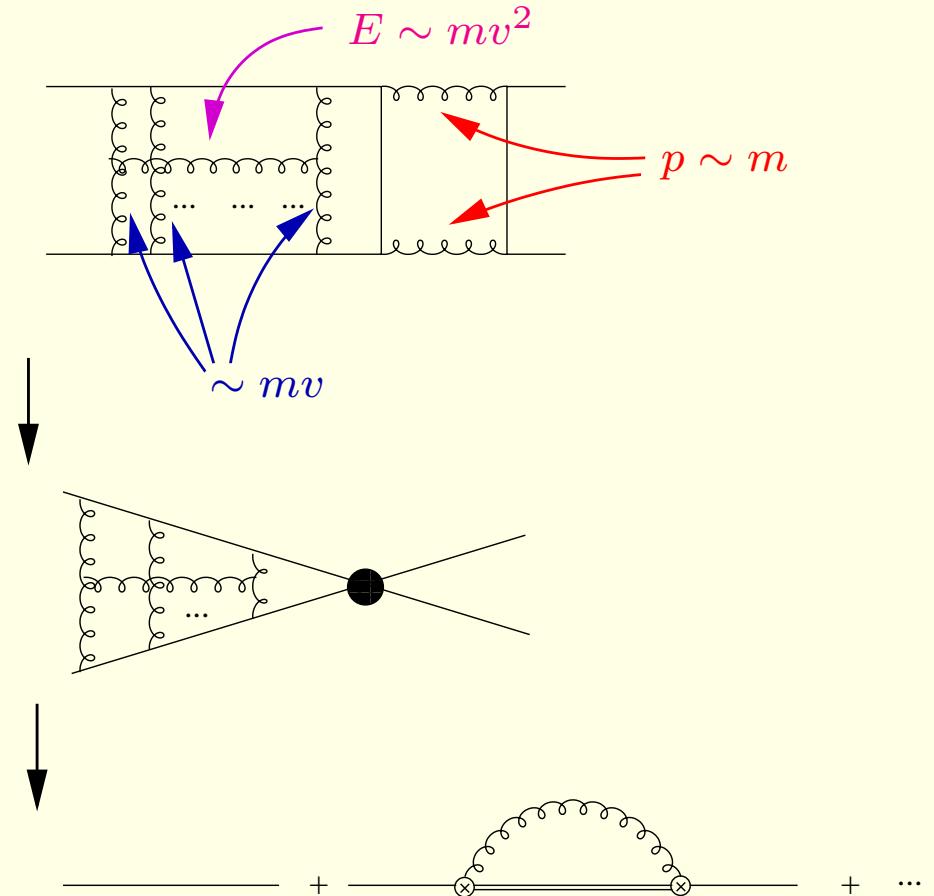
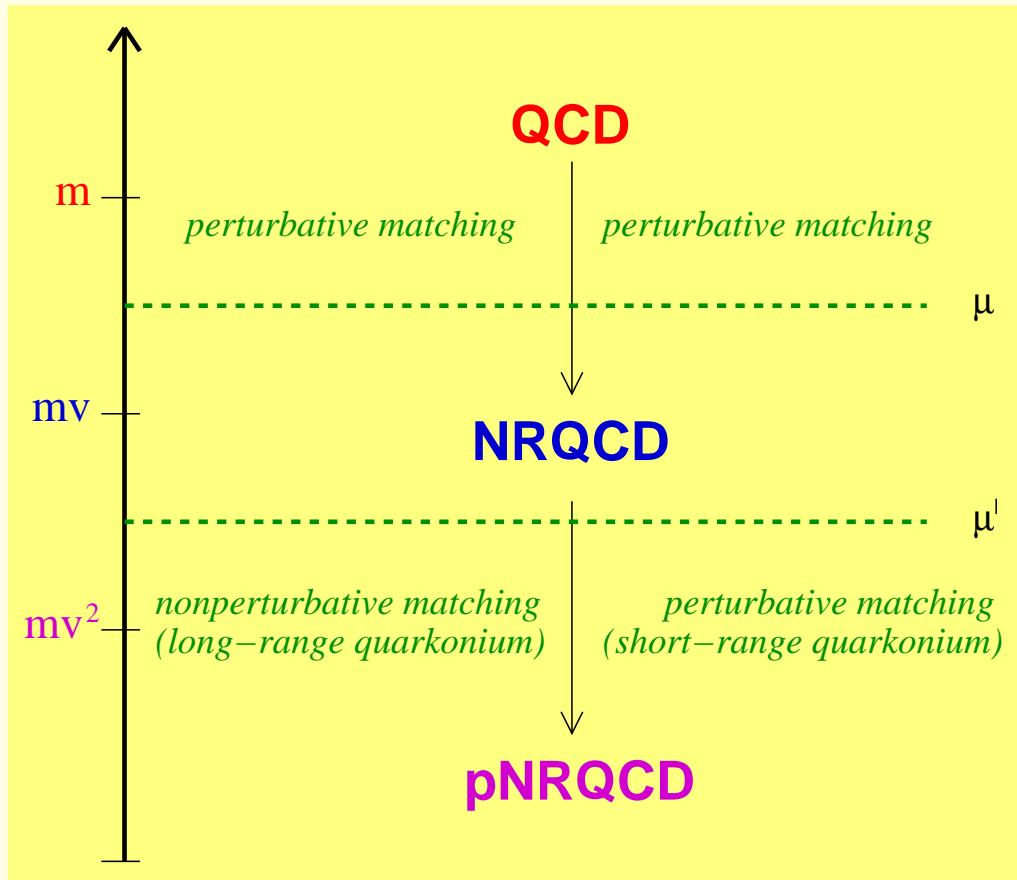
- The system is **non-relativistic**: $p \sim mv$ and $E = \frac{p^2}{m} + V \sim mv^2$.

Non-relativistic scales in QCD

Scales get entangled.

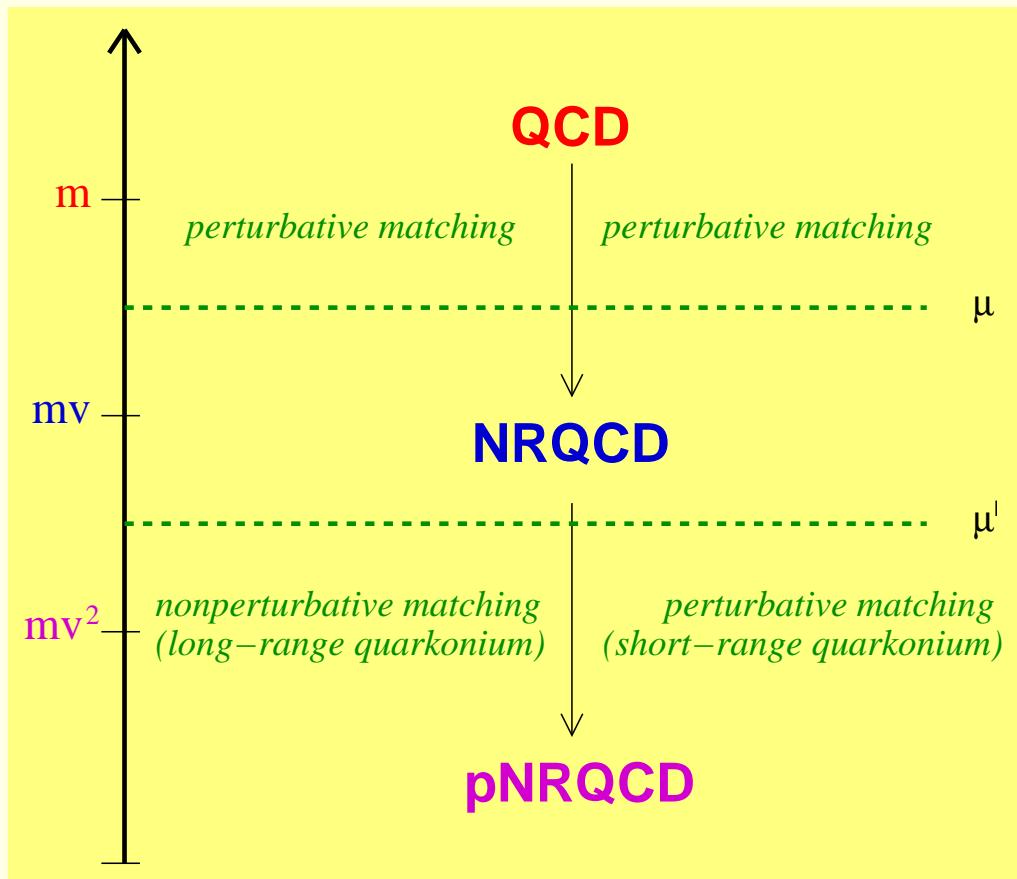


EFTs for systems made of two heavy quarks



- They exploit the expansion in v / factorization of low and high energy contributions.
- They are renormalizable order by order in v .
- In perturbation theory (PT), RG techniques provide resummation of large logs.

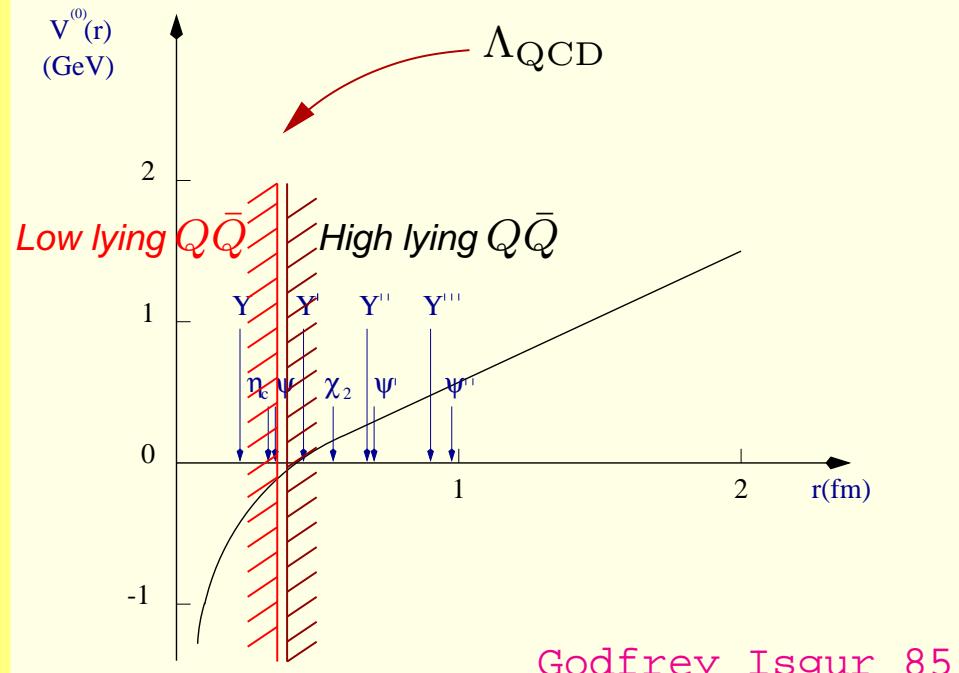
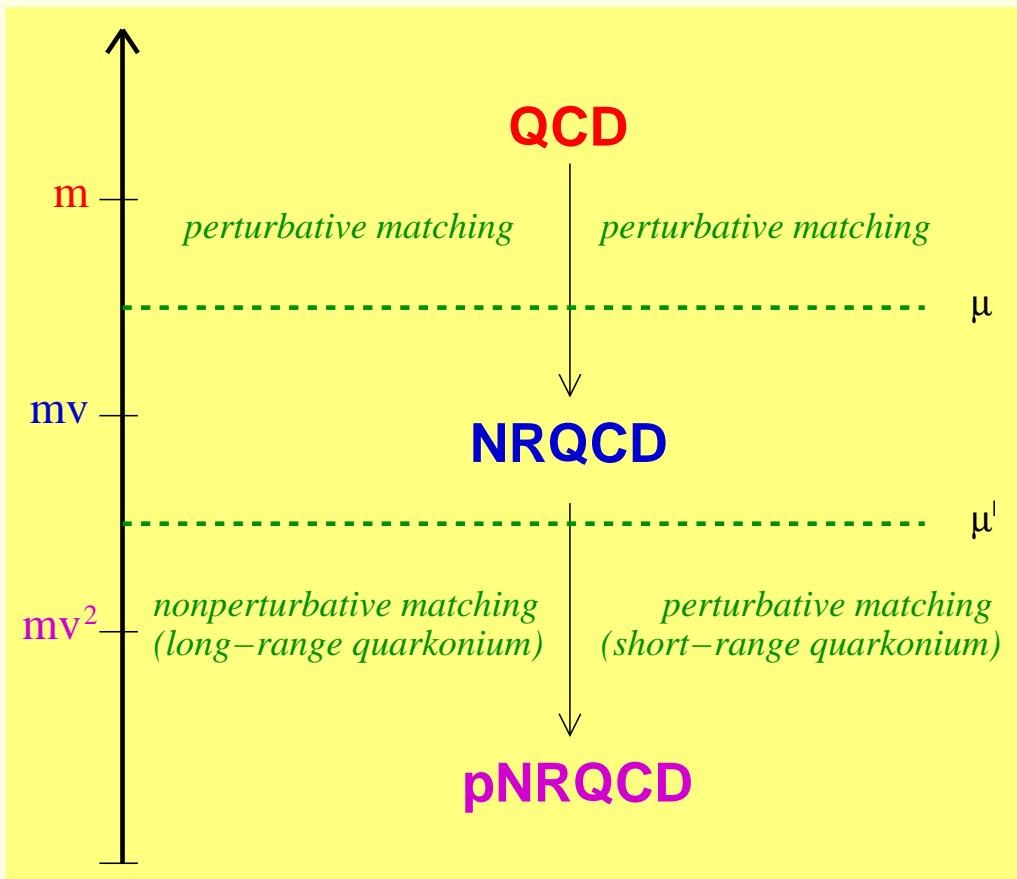
EFTs for systems made of two heavy quarks



Caswell Lepage 86, Lepage Thacker 88
Bodwin Braaten Lepage 95, ...

Pineda Soto 97, Brambilla et al 99
Kniehl et al 99, ...
Luke Manohar 97, Luke Savage 98
Labelle 98, Grinstein Rothstein 98
Griesshammer 98, Luke et al 00
Hoang 02, ... → vNRQCD

EFTs for systems made of two heavy quarks



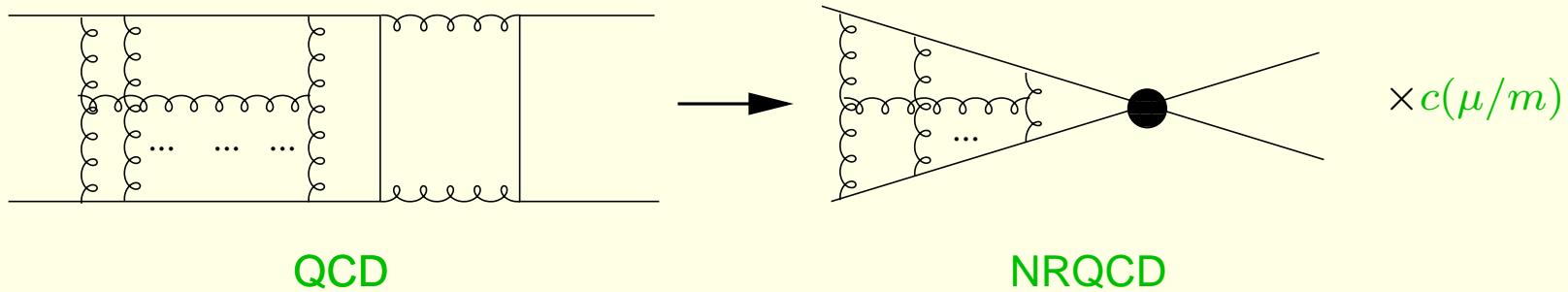
Godfrey Isgur 85

A potential picture arises at the level of pNRQCD:

- the potential is perturbative if $mv \gg \Lambda_{\text{QCD}}$
- the potential is non-perturbative if $mv \sim \Lambda_{\text{QCD}}$

NRQCD

NRQCD is obtained by integrating out modes associated with the scale m



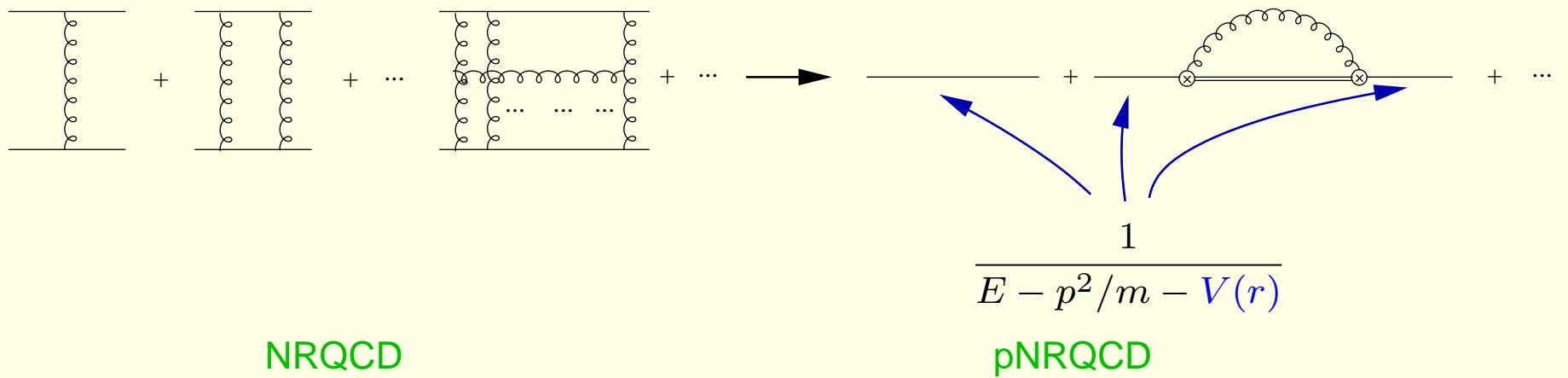
- The matching is perturbative.
- The Lagrangian is organized as an expansion in $1/m$ and $\alpha_s(m)$:

$$\mathcal{L}_{\text{NRQCD}} = \sum_n c(\alpha_s(m/\mu)) \times O_n(\mu, \lambda)/m^n$$

Suitable to describe annihilation and production of quarkonium.

pNRQCD

pNRQCD is obtained by integrating out modes associated with the scale $\frac{1}{r} \sim mv$



- The Lagrangian is organized as an expansion in $1/m$, r , and $\alpha_s(m)$:

$$\mathcal{L}_{\text{pNRQCD}} = \sum_k \sum_n \frac{1}{m^k} \times c_k(\alpha_s(m/\mu)) \times V(r\mu', r\mu) \times O_n(\mu', \lambda) r^n$$

pNRQCD for $mv \gg \Lambda_{\text{QCD}}$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \text{Tr} \left\{ \textcolor{magenta}{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{m} - \textcolor{green}{V}_s - \frac{V_s^{(1)}}{m} + \dots \right) \textcolor{magenta}{S} \right.$$

$$\left. + \textcolor{magenta}{O}^\dagger \left(iD_0 - \frac{\mathbf{p}^2}{m} - \textcolor{green}{V}_o - \frac{V_o^{(1)}}{m} + \dots \right) \textcolor{magenta}{O} \right\}$$

LO in $\textcolor{green}{r}$

$$+ \textcolor{green}{V}_A \text{Tr} \left\{ \textcolor{magenta}{O}^\dagger \mathbf{r} \cdot g \mathbf{E} \textcolor{magenta}{S} + \textcolor{magenta}{S}^\dagger \mathbf{r} \cdot g \mathbf{E} \textcolor{magenta}{O} \right\}$$

$$+ \frac{\textcolor{green}{V}_B}{2} \text{Tr} \left\{ \textcolor{magenta}{O}^\dagger \mathbf{r} \cdot g \mathbf{E} \textcolor{magenta}{O} + \textcolor{magenta}{O}^\dagger \mathbf{O} \mathbf{r} \cdot g \mathbf{E} \right\}$$

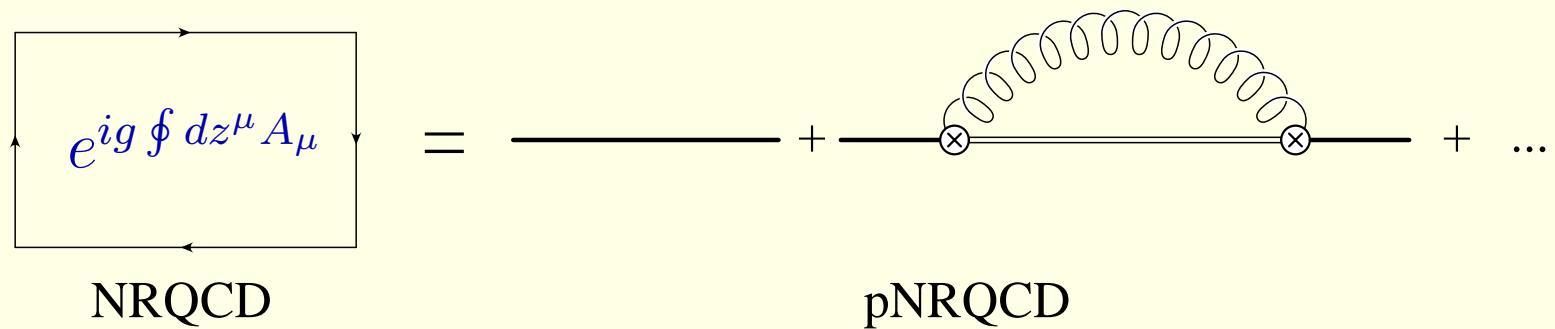
$$+ \dots$$

NLO in $\textcolor{green}{r}$

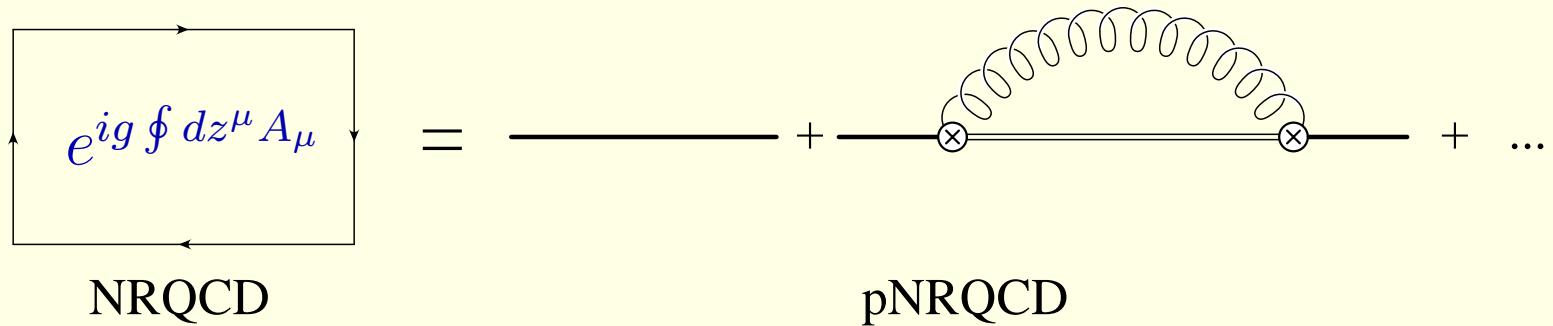
The potential is a Wilson coefficient of an EFT. In general, it undergoes renormalization, develops scale dependence and satisfies renormalization group equations, which allow to resum large logarithms.

2. Calculation

The Static Potential



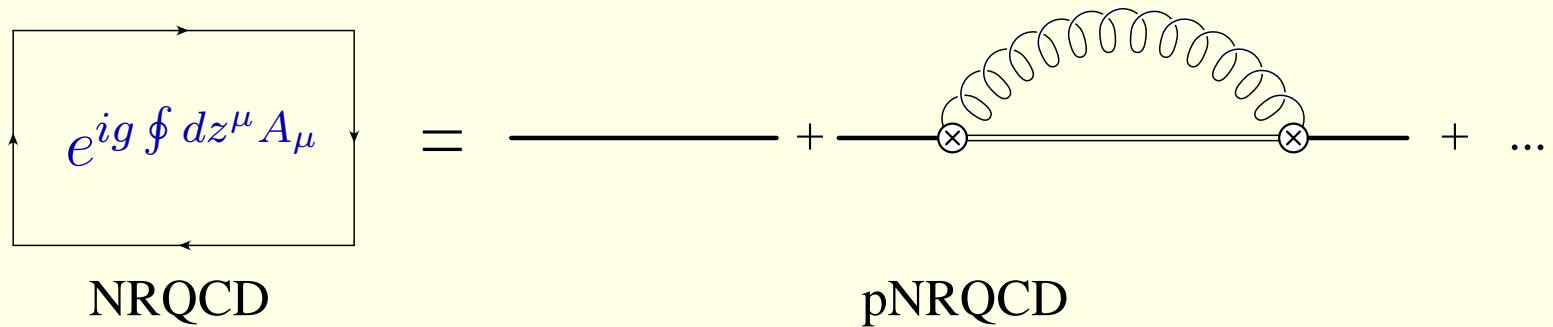
The Static Potential



$$\lim_{T \rightarrow \infty} \frac{i}{T} \ln \langle \boxed{} \rangle = V_s(\mathbf{r}, \mu) - i \frac{g^2}{N_c} V_A^2 \int_0^\infty dt e^{-it(V_o - V_s)} \langle \text{Tr}(\mathbf{r} \cdot \mathbf{E}(t) \mathbf{r} \cdot \mathbf{E}(0)) \rangle(\mu) + \dots$$

ultrasoft contribution

The Static Potential



$$\lim_{T \rightarrow \infty} \frac{i}{T} \ln \langle \boxed{} \rangle = V_s(\mathbf{r}, \mu) - i \frac{g^2}{N_c} V_A^2 \int_0^\infty dt e^{-it(V_o - V_s)} \langle \text{Tr}(\mathbf{r} \cdot \mathbf{E}(t) \mathbf{r} \cdot \mathbf{E}(0)) \rangle(\mu) + \dots$$

ultrasoft contribution

* The μ dependence cancels between the two terms in the right-hand side:

$$V_s \sim \ln r\mu, \ln^2 r\mu, \dots$$

$$\text{ultrasoft contribution} \sim \ln(V_o - V_s)/\mu, \ln^2(V_o - V_s)/\mu, \dots \ln r\mu, \ln^2 r\mu, \dots$$

Static Wilson loop

$$\lim_{T \rightarrow \infty} \frac{i}{T} \ln \langle \boxed{} \rangle = -C_F \frac{\alpha_s(1/r)}{r} \left[1 + a_1 \frac{\alpha_s(1/r)}{4\pi} + a_2 \left(\frac{\alpha_s(1/r)}{4\pi} \right)^2 + \dots \right]$$

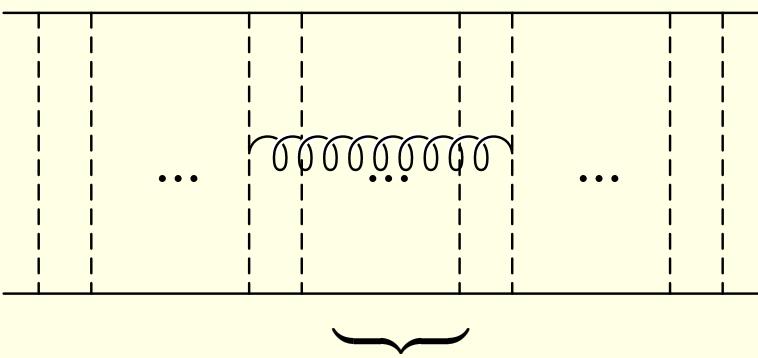
is known at two loops:

$$a_1 = \frac{31}{9} C_A - \frac{10}{9} n_f + 2\gamma_E \beta_0,$$

Billoire 80

$$\begin{aligned} a_2 &= \left(\frac{4343}{162} + 4\pi^2 - \frac{\pi^4}{4} + \frac{22}{3} \zeta(3) \right) C_A^2 - \left(\frac{899}{81} + \frac{28}{3} \zeta(3) \right) C_A n_f \\ &\quad - \left(\frac{55}{6} - 8\zeta(3) \right) C_F n_f + \frac{100}{81} n_f^2 + 4\gamma_E \beta_0 a_1 + \left(\frac{\pi^2}{3} - 4\gamma_E^2 \right) \beta_0^2 + 2\gamma_E \beta_1 \end{aligned}$$

Appelquist–Dine–Muzinich diagrams


$$= -\frac{C_F C_A^3}{12} \frac{\alpha_s}{r} \frac{\alpha_s^3}{\pi} \ln \left[\frac{C_A \alpha_s}{2r} \times r \right]$$
$$\sim \exp(-i(V_o - V_s) T)$$

Appelquist Dine Muzinich 78, Brambilla Pineda Soto Vairo 99

Static octet potential

$$\lim_{T \rightarrow \infty} \frac{i}{T} \ln \frac{\langle \text{Diagram} \rangle}{\langle \phi_{ab}^{\text{adj}} \rangle} = \frac{1}{2N_c} \frac{\alpha_s(1/r)}{r} \left[1 + b_1 \frac{\alpha_s(1/r)}{4\pi} + b_2 \left(\frac{\alpha_s(1/r)}{4\pi} \right)^2 + \dots \right]$$

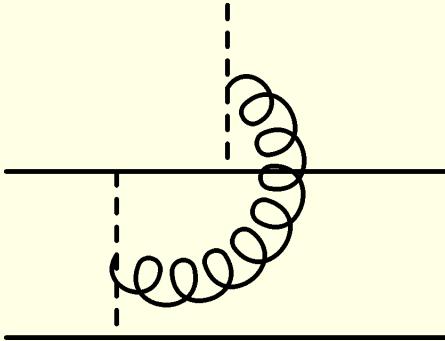
Is known at two loops.

$$b_1 = a_1$$

$$b_2 = a_2 + C_A^2 (\pi^4 - 12\pi^2)$$

$$V_A$$

The first contributing diagrams are of the type:

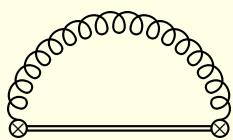


Therefore

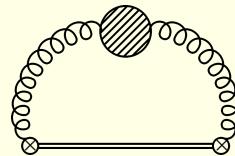
$$V_A(r, \mu) = 1 + \mathcal{O}(\alpha_s^2)$$

Chromoelectric field correlator: $\langle E(t)E(0) \rangle$

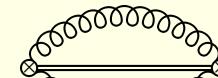
Is known at NLO.



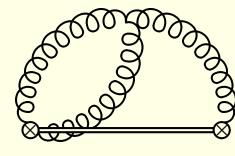
LO



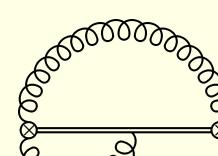
(a)



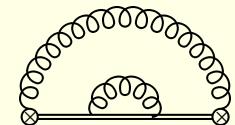
(b)



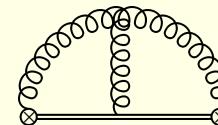
(c)



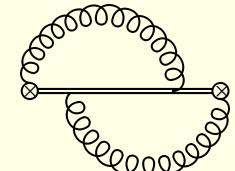
(d)



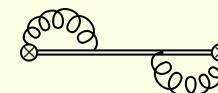
(e)



(f)



(g)



(h)

NLO

Static singlet potential

$$\begin{aligned}
 V_s(r, \mu) = & -C_F \frac{\alpha_s(1/r)}{r} \left[1 + a_1 \frac{\alpha_s(1/r)}{4\pi} + a_2 \left(\frac{\alpha_s(1/r)}{4\pi} \right)^2 \right. \\
 & + \left(\frac{16\pi^2}{3} C_A^3 \ln r\mu + a_3 \right) \left(\frac{\alpha_s(1/r)}{4\pi} \right)^3 \\
 & \left. + \left(a_4^{L2} \ln^2 r\mu + \left(a_4^L + \frac{16}{9}\pi^2 C_A^3 \beta_0 (-5 + 6 \ln 2) \right) \ln r\mu + a_4 \right) \left(\frac{\alpha_s(1/r)}{4\pi} \right)^4 \right]
 \end{aligned}$$

$$\begin{aligned}
 a_4^{L2} &= -\frac{16\pi^2}{3} C_A^3 \beta_0 \\
 a_4^L &= 16\pi^2 C_A^3 \left[a_1 + 2\gamma_E \beta_0 + n_f \left(-\frac{20}{27} + \frac{4}{9} \ln 2 \right) \right. \\
 &\quad \left. + C_A \left(\frac{149}{27} - \frac{22}{9} \ln 2 + \frac{4}{9} \pi^2 \right) \right]
 \end{aligned}$$

Static singlet potential

$$\begin{aligned} V_s(r, \mu) = & -C_F \frac{\alpha_s(1/r)}{r} \left[1 + a_1 \frac{\alpha_s(1/r)}{4\pi} + a_2 \left(\frac{\alpha_s(1/r)}{4\pi} \right)^2 \right. \\ & + \left(\frac{16\pi^2}{3} C_A^3 \ln r\mu + a_3 \right) \left(\frac{\alpha_s(1/r)}{4\pi} \right)^3 \\ & \left. + \left(a_4^{L2} \ln^2 r\mu + \left(a_4^L + \frac{16}{9}\pi^2 C_A^3 \beta_0 (-5 + 6 \ln 2) \right) \ln r\mu + a_4 \right) \left(\frac{\alpha_s(1/r)}{4\pi} \right)^4 \right] \end{aligned}$$

- The logarithmic contribution at N^3LO may be extracted from the one-loop calculation of the ultrasoft contribution;
- the single logarithmic contribution at N^4LO may be extracted from the two-loop calculation of the ultrasoft contribution.

Static singlet potential

$$\begin{aligned}
V_s(r, \mu) = & -C_F \frac{\alpha_s(1/r)}{r} \left[1 + a_1 \frac{\alpha_s(1/r)}{4\pi} + a_2 \left(\frac{\alpha_s(1/r)}{4\pi} \right)^2 \right. \\
& + \left(\frac{16\pi^2}{3} C_A^3 \ln r\mu + a_3 \right) \left(\frac{\alpha_s(1/r)}{4\pi} \right)^3 \\
& \left. + \left(a_4^{L2} \ln^2 r\mu + \left(a_4^L + \frac{16}{9}\pi^2 C_A^3 \beta_0 (-5 + 6 \ln 2) \right) \ln r\mu + a_4 \right) \left(\frac{\alpha_s(1/r)}{4\pi} \right)^4 \right]
\end{aligned}$$

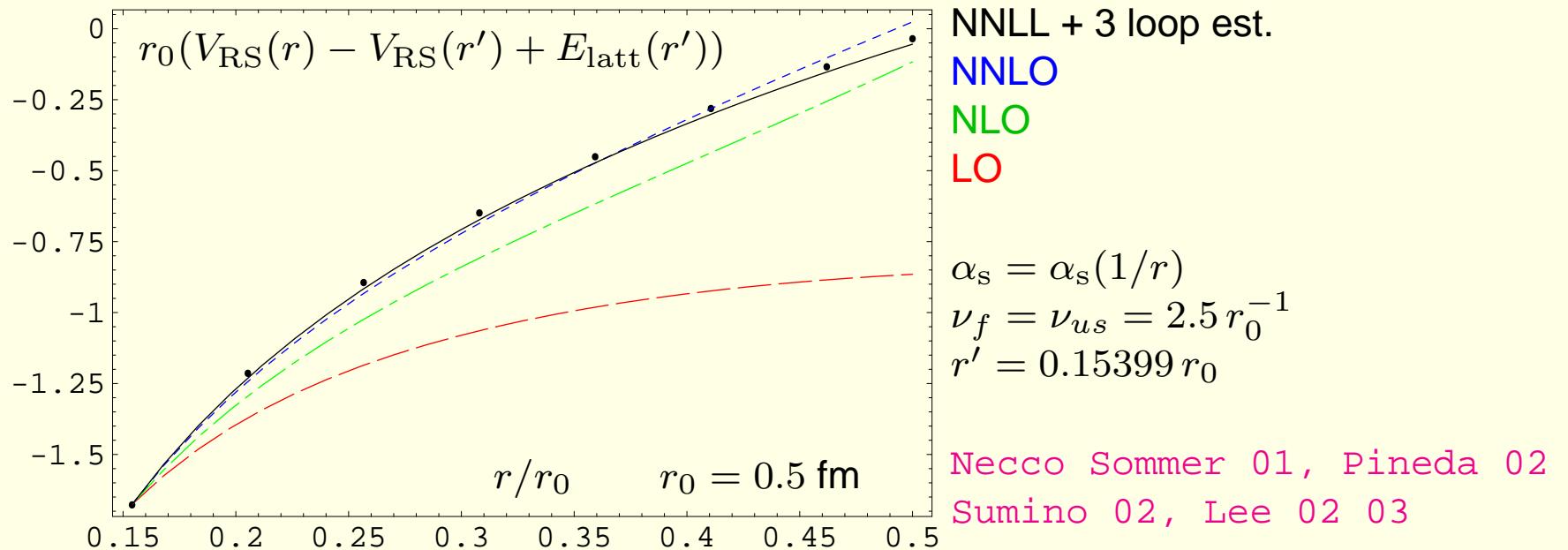
The potential is a Wilson coefficient of an EFT. In general, it undergoes renormalization, develops scale dependence and satisfies renormalization group equations, which allow to resum large logarithms. It carries also large $(\alpha_s \beta_0)^n$ contributions of the renormalon type.

3. Applications

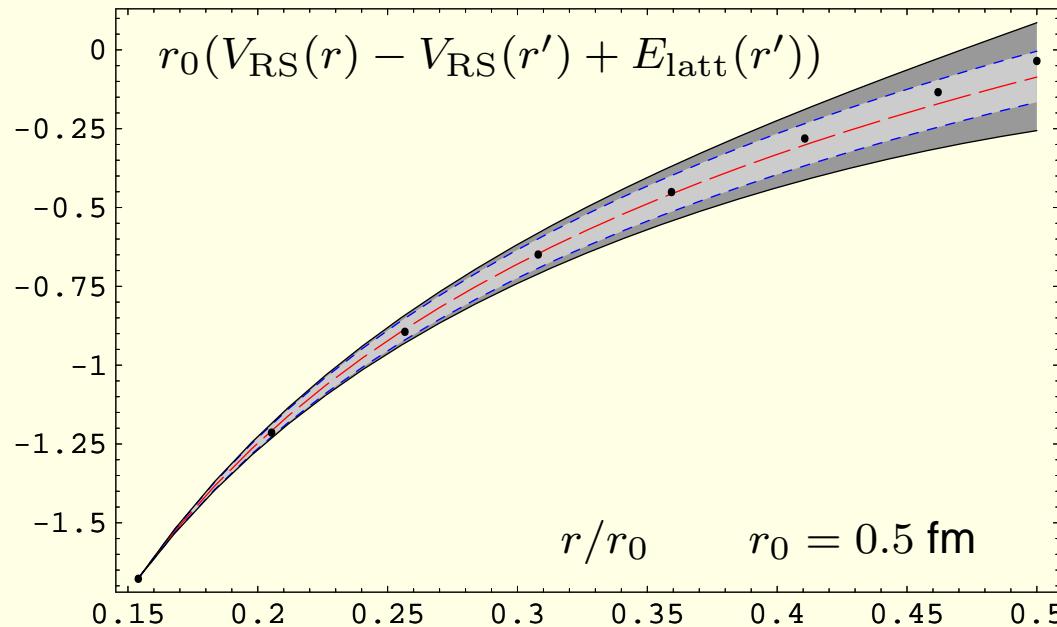
Static energy

$$\begin{aligned} E_0(r) = & -\frac{C_F \alpha_s(1/r)}{r} \left\{ 1 + \frac{\alpha_s(1/r)}{4\pi} [a_1 + 2\gamma_E \beta_0] \right. \\ & + \left(\frac{\alpha_s(1/r)}{4\pi} \right)^2 \left[a_2 + \left(\frac{\pi^2}{3} + 4\gamma_E^2 \right) \beta_0^2 + \gamma_E (4a_1\beta_0 + 2\beta_1) \right] \\ & + \left(\frac{\alpha_s(1/r)}{4\pi} \right)^3 \left[\frac{16\pi^2}{3} C_A^3 \ln \frac{C_A \alpha_s(1/r)}{2} + \tilde{a}_3 \right] \\ & + \left(\frac{\alpha_s(1/r)}{4\pi} \right)^4 \left[a_4^{L2} \ln^2 \frac{C_A \alpha_s(1/r)}{2} + a_4^L \ln \frac{C_A \alpha_s(1/r)}{2} + \tilde{a}_4 \right] \\ & \left. + \dots \right\} \end{aligned}$$

Static energy vs lattice QCD



Static energy vs lattice QCD



Pineda 02

No signal of short-range linear non-perturbative effects.

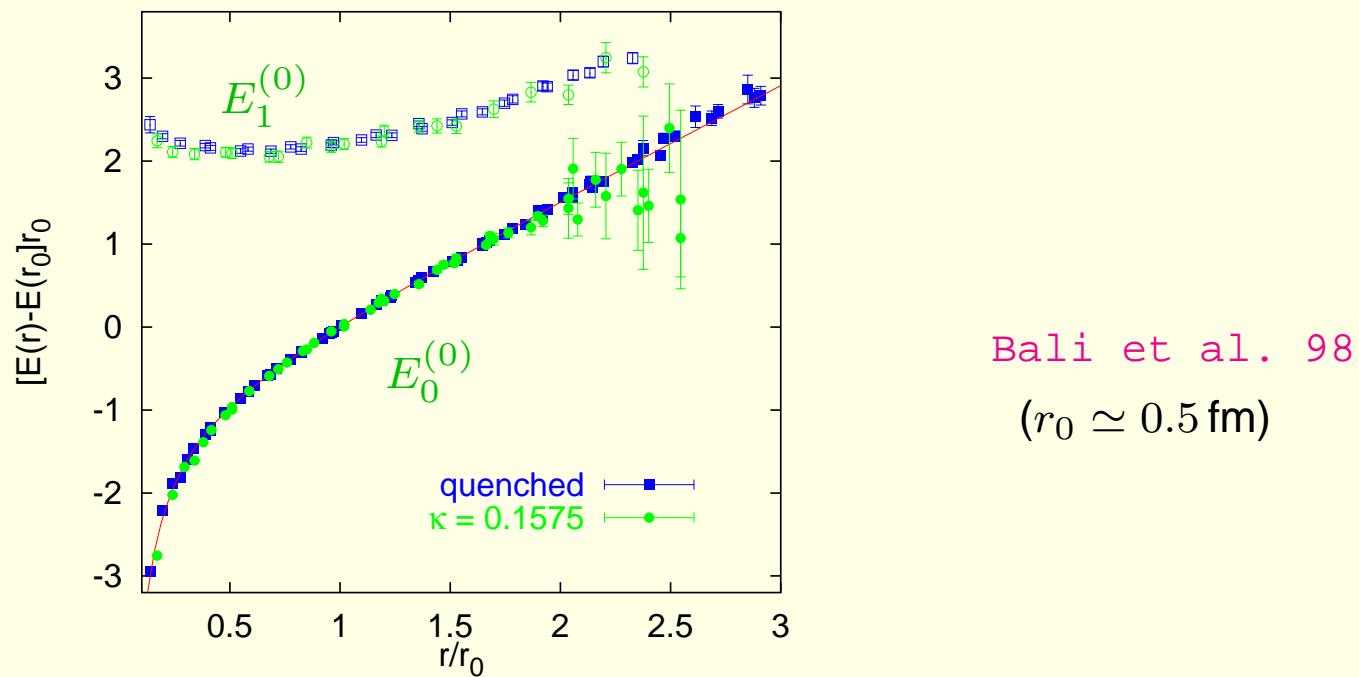
Applications to quarkonium ground-state observables

- Heavy quark masses at NNLO and NNLL*.
- $\alpha_s(M_{\Upsilon(1S)})$ at NLO.
- B_c mass at NNLO.
- Hyperfine splittings in charmonium, B_c and bottomonium at NLL.
- $\Gamma(\eta_b \rightarrow \gamma\gamma)/\Gamma(\Upsilon(1S) \rightarrow e^+e^-)$ at NNLL.
- $\Gamma(\Upsilon(1S) \rightarrow \gamma X)$ at NLO.
- Magnetic dipole transitions at NNLO.
- $e^+e^- \rightarrow t\bar{t}$ cross section near threshold at NNLL*.
- ...

4. Non-perturbative potential

pNRQCD for $mv \sim \Lambda_{\text{QCD}}$

- All scales above mv^2 are integrated out.
- All gluonic excitations between heavy quarks are integrated out since they develop a gap of order Λ_{QCD} with the static $Q\bar{Q}$ energy.



pNRQCD for $mv \sim \Lambda_{\text{QCD}}$

- All scales above mv^2 are integrated out.
 - All gluonic excitations between heavy quarks are integrated out since they develop a gap of order Λ_{QCD} with the static $Q\bar{Q}$ energy.
- ⇒ The singlet quarkonium field S of energy mv^2 is the only the degree of freedom of pNRQCD (up to ultrasoft light quarks, e.g. pions).

pNRQCD for $mv \sim \Lambda_{\text{QCD}}$

$$\mathcal{L} = \text{Tr} \left\{ \textcolor{red}{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{\textcolor{blue}{m}} - \textcolor{green}{V}_s \right) \textcolor{red}{S} \right\}$$

Brambilla Pineda Soto Vairo 00

pNRQCD for $mv \sim \Lambda_{\text{QCD}}$

$$\mathcal{L} = \text{Tr} \left\{ \textcolor{magenta}{S}^\dagger \left(i\partial_0 - \frac{\mathbf{p}^2}{\textcolor{blue}{m}} - \textcolor{green}{V}_s \right) \textcolor{magenta}{S} \right\}$$

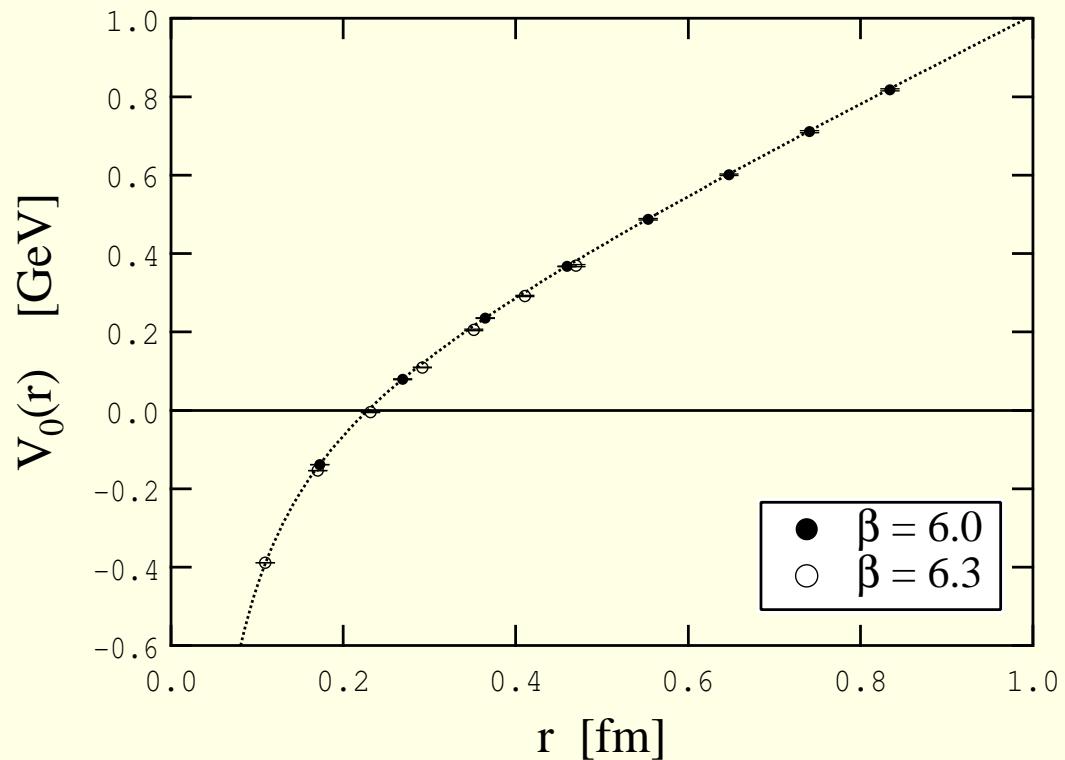
Brambilla Pineda Soto Vairo 00

- The potential V_s ($\text{Re } V_s + i \text{Im } V_s$) is non-perturbative:
 - (a) to be determined from the lattice;
 - (b) to be determined from QCD vacuum models.

Creutz et al. 82, Campostrini 85, Michael 85, Born et al. 94,
Bali et al 97, Bali 00, Koma et al 06, Brambilla et al. 93, 95, 97, 98

Static potential

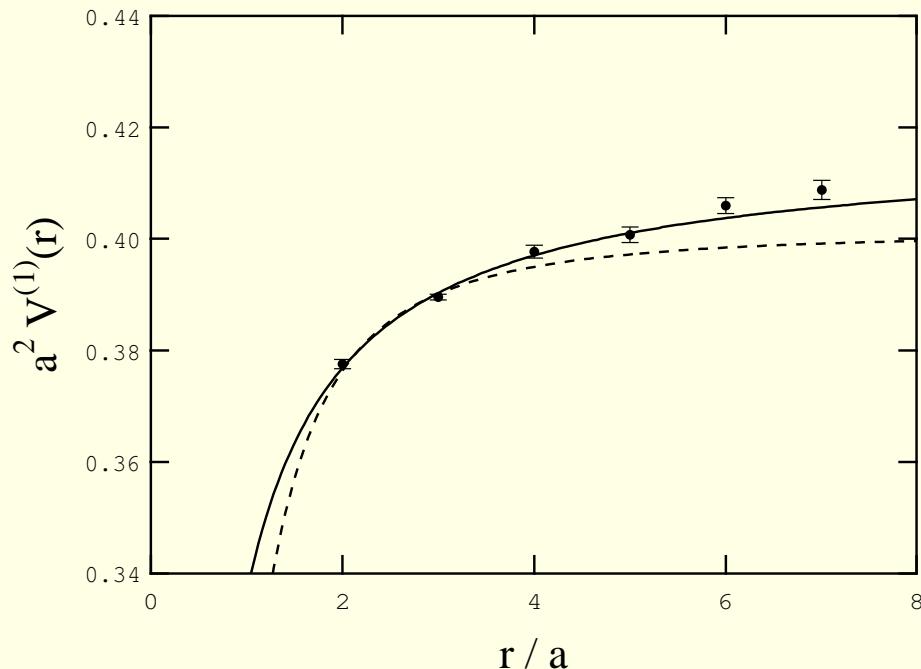
$$V_s^{(0)} = \lim_{T \rightarrow \infty} \frac{i}{T} \ln \langle \boxed{\quad} \rangle$$



1/m potential

$$\frac{V_s^{(1)}}{m} = -\frac{1}{2m} \int_0^\infty dt t \langle \boxed{\text{E}} \rangle$$

Brambilla et al 00



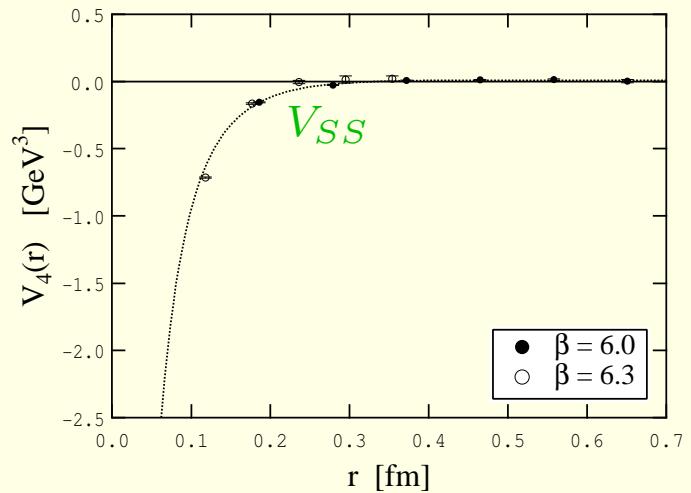
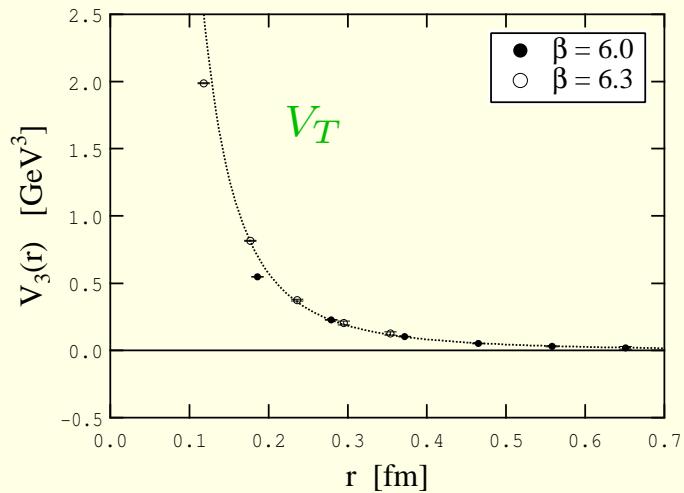
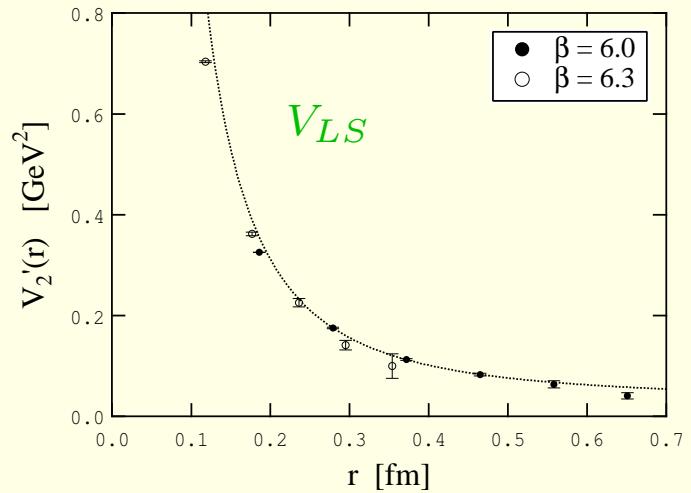
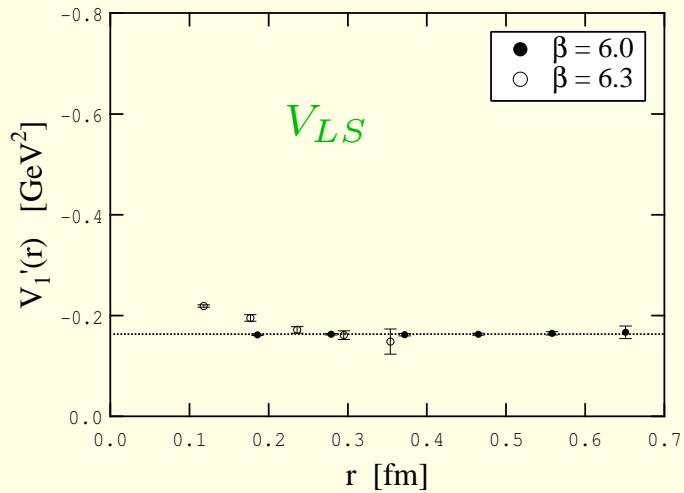
$$V^{(1)} = -\frac{c}{r} + d$$

$\frac{2c}{m_c} \frac{1}{r} \approx \frac{1}{r}$ part of the static potential

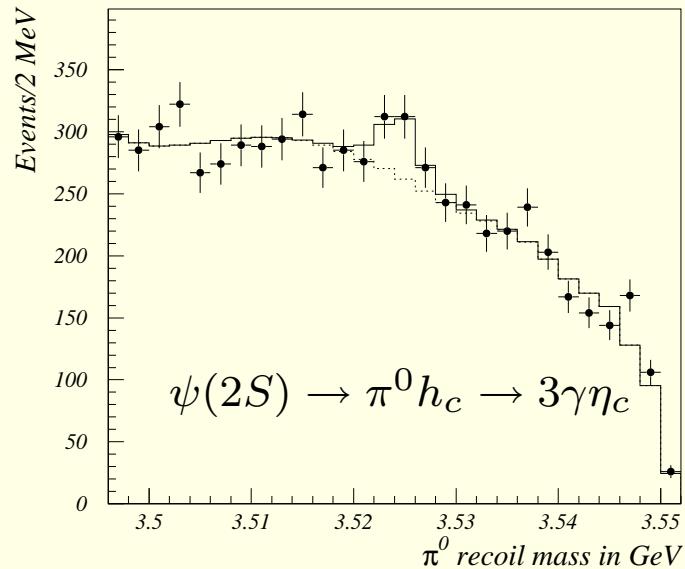
$\frac{2c}{m_b} \frac{1}{r} \approx 26\%$ of the $\frac{1}{r}$ part of the static potential

Koma Koma Wittig 06

Spin-dependent potentials



h_c



$$M = 3524.4 \pm 0.6 \pm 0.4 \text{ MeV}$$

CLEO 05

Also

$$M = 3525.8 \pm 0.2 \pm 0.2 \text{ MeV}, \quad \Gamma < 1 \text{ MeV}$$

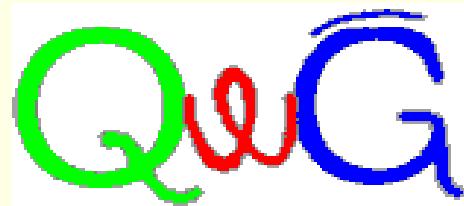
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- To be compared with $M_{c.o.g.}(1P) = 3525.36 \pm 0.2 \pm 0.2 \text{ MeV}$.

Conclusions

Non-relativistic EFTs provide a rigorous definition of the potential between two heavy quarks.

- In the perturbative regime, it is a key ingredient for precision calculations of several threshold observables.
- In the non-perturbative regime, it may be calculated on the lattice. Embedded in an EFT it provides an alternative to more traditional lattice calculations with heavy quarks (e.g. NRQCD).



<http://www.qwg.to.infn.it>

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