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Signal-to-noise ratio for a stochastic background of massive relic particles

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Abstract

We estimate the signal-to-noise ratio for two gravitational detectors interacting with a stochastic background of massive scalar waves. We find that the present experimental level of sensitivity could be already enough to detect a signal from a light but non-relativistic component of dark matter, even if the coupling is weak enough to exclude observable deviations from standard gravitational interactions, provided the mass is not too far from the sensitivity and overlapping band of the two detectors.

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The sensitivity of present detectors to a stochastic background of relic gravitational waves has been recently discussed in detail in many papers (see [1] - [4], for instance, and references therein). The sensitivity analysis has also been extended to include scalar waves [5], and scalar stochastic backgrounds [6] of massless (or massive, but light enough) scalar particles, interacting with gravitational strength with the detectors. At present, however, no analysis seems to be available on the possible response of the gravitational antennas to a scalar stochastic background of *non-relativistic* particles.

The aim of this paper is to compute the signal-to-noise ratio (SNR) for a pair of gravitational antennas by taking into account the possible mass of the background particles, in order to discuss in some detail the possible effects of the non-relativistic branch of their spectrum.

We shall consider a cosmic stochastic background of massive scalar waves, whose energy density is coupled to the total mass of the detector with gravitational strength (or weaker). We shall assume that the background is characterized by a spectral energy density $\Omega(p) = d(\rho/\rho_c)/d\ln p$, which we measure in units of critical density $\rho_c = 3H_0^2 M_p^2/8\pi$, and which extends in momentum space from $p = 0$ to $p = p_1$ (p_1 is a cut-off scale depending on the details of the production mechanism). As a function of the frequency $f = E(p) = (m^2 + p^2)^{1/2}$, the spectrum $\tilde{\Omega}(f)$,

$$\tilde{\Omega}(f) \equiv \frac{d(\rho/\rho_c)}{d\ln f} = \left(\frac{f}{p}\right)^2 \Omega(p) \tag{1}$$

thus extends over frequencies $f \geq m$, from $f = m$ to $f = f_1 = (m^2 + p_1^2)^{1/2}$ (note that we are using "unconventional" units in which $h = 1$, for a better comparison with the observable quantities used in the experimental analysis of gravitational antennas). We may thus distinguish three phenomenological possibilities.

- $m \gg f_0$, where f_0 is any frequency in the sensitivity band of the detector (tipically, if we are considering resonant masses and interferometers, $f_0 \sim 10^2 - 10^3$ Hz). In this case we expect no signal, as the response to the background should be totally suppressed by the intrinsic noise of the detector.

- $m \ll f_0$. In this case the detector, in its sensitivity band, responds to a relativistic frequency spectrum, and the SNR can be easily estimated by using the standard results. For a relativistic background of cosmological origin, however, the maximal amplitude allowed by nucleosynthesis [7] is $\Omega \sim 10^{-5}$, possibly suppressed by a factor $q^2 \ll 1$ (in the interaction with the antenna) to avoid scalar-induced, long-range violations of the equivalence principle (see [8], for instance). We thus expect from such a scalar background a response not larger than from a background of relic gravitons, and then too weak for the sensitivity of present detectors.

- $m \sim f_0$. In this case the mass is the frequency band of maximal sensitivity, and the detector can respond *resonantly* also to the *non-relativistic part* of the background (i.e. to the branch $p < m$ of $\Omega(p)$). In the non-relativistic sector, on the other hand, the background amplitude is not constrained by the nucleosynthesis bound, because the non-relativistic energy density grows in time with respect to the relativistic one: it could be sub-dominant at the nucleosynthesis epoch, even if today has reached a near-to-critical amplitude $\Omega \sim 1$ (i.e., even if the massive background we are considering represents today a significant fraction of the cosmological dark matter). In such case, it will be shown in this paper that the present sensitivity of the existing gravitational antennas could be enough to distinguish the physical signal from the intrinsic experimental noise.

We will follow the standard approach (see [2], for instance) in which the outputs of two detectors, $s_i(t)$, $i = 1, 2$, are correlated over an integration time T, to define a signal:

$$S = \int_{-T/2}^{T/2} dt\ dt' s_1(t) s_2(t') Q(t - t'). \tag{2}$$

Here $Q(t)$ is a real "filter" function, determined so as to optimize the signal-to-noise ratio (SNR), defined by an ensemble average as:

$$SNR = \langle S \rangle / \Delta S \equiv \langle S \rangle \left(\langle S^2 \rangle - \langle S \rangle^2 \right)^{-1/2} \tag{3}$$

The outputs $s_i(t) = h_i(t) + n_i(t)$ contain the physical strain induced by the cosmic background, h_i, and the intrinsic instrumental noise, n_i. The two noises are supposed to be uncorrelated (i.e., statistically independent), $\langle n_1(t) n_2(t') \rangle = 0$, and much larger in magnitude than the physical strains h_i. Also, the cosmic background is assumed to be isotropic, stationary and Gaussian, with $\langle h_i \rangle = 0$. It follows that:

$$\langle S \rangle = \int_{-T/2}^{T/2} dt\ dt' \langle h_1(t) h_2(t') \rangle Q(t - t'). \tag{4}$$

An explicit compuation of the strain, at this point, would require a specific model of the interaction between the scalar background and the detector. We will assume in this paper that the strain $h_i(t)$, like in the case of gravitational waves [2] and Brans-Dicke scalars [6], varies in time like the scalar fluctuation $\phi(x_i, t)$ perturbing the detector (computed at the detector position $x = x_i$), and is proportional to the so-called "pattern function" $F_i(\hat{n}) = e_{ab}(\hat{n}) D_i^{ab}$, where $\hat{n}$ is a unit vector specifying a direction on the two sphere, $e_{ab}(\hat{n})$ is the polarization tensor of the scalar along $\hat{n}$, and D_i^{ab} is the detector tensor, specifying the orientation of the arms of the i-th detector.

The field $\phi(x,t)$ may represent the scalar (i.e, zero helicity) component of the metric fluctuations generated by the scalar component of the background, as in [6], or could even represent the background field itself, directly coupled to the detector through a "scalar charge" q_i (for instance, a dilatonic charge), as discussed in [9]. To take into account this second possibility, we shall explicitly introduce the scalar charge in the strain, by setting

$$h_i(t) = q_i \phi(x_i, t) e_{ab}(\hat{n}) D_i^{ab}, \tag{5}$$

where $q_i = 1$ for scalar metric fluctuations, and $q_i < 1$ for long-range scalar fields, phenomenologically constrained by the gravitational tests. The dimensionless parameter q_i represents the net scalar charge per unit of gravitational mass of the detector, and is in general composition-dependent [9].

To compute the average signal (4) we now expand the strain in momentum space,

$$\begin{aligned} & h_i(t) = q_i \int dp \int d^2\hat{n}\ \phi(p,\hat{n}) F_i(\hat{n}) e^{2\pi i [p\hat{n}\cdot\vec{x}_i - E(p)t]}, \\ & p = |\vec{p}|, \qquad \vec{p}/p = \hat{n}, \qquad E(p) = f = (m^2 + p^2)^{1/2}, \end{aligned} \tag{6}$$

($d^2\hat{n}$ denotes the angular integral over the unit two-sphere), and we use the stochastic condition

$$\langle \phi^\star(p,\hat{n}), \phi(p',\hat{n}') \rangle = \delta(p-p')\delta^2(\hat{n}-\hat{n}')\Phi(p). \tag{7}$$

The isotropic function $\Phi(p)$ can be expressed in terms of the spectral energy density $\Omega(p)$, defined by

$$\rho = \rho_c \int d\ln p\ \Omega(p) = \frac{M_P^2}{16\pi} \langle |\dot{\phi}|^2 \rangle, \tag{8}$$

(M_P is the Planck mass) from which:

$$\Phi(p) = \frac{3H_0^2 \Omega(p)}{8\pi^3 p E^2(p)}. \tag{9}$$

By inserting the momentum expansion into eq. (4), and assuming, as usual, that the observation time T is much larger than the typical time intervals $t - t'$ for which $Q \neq 0$, we finally obtain:

$$\langle S \rangle = q_1 q_2 T \frac{2H_0^2}{5\pi^2} \int \frac{dp}{pE^2(p)} \gamma(p) Q(p) \Omega(p). \tag{10}$$

We have defined the overlap function $\gamma(p)$ and the filter function $Q(p)$, in momentum space, as follows:

$$\gamma(p)=\frac{15}{16\pi}\int d^2\hat{n}F_1(\hat{n})F_2(\hat{n})e^{2\pi ip\hat{n}\cdot(\vec{x}_2-\vec{x}_1)},$$
$$Q(p)=\int dt' Q(t-t')e^{2\pi iE(p)(t-t')}. \tag{11}$$

Note that the overlap function depends on the relative distance of the two gravitational antennas and on their particular geometric configuration. In the above equation, in particular, $\gamma(p)$ has been normalized to the response of an interferometric detector to a scalar wave [6].

We need now to compute the variance ΔS^2 which, for uncorrelated noises, much larger than the physical strains, can be expressed as [2]:

$$\Delta S^2 \simeq \langle S^2\rangle=\int_{-T/2}^{T/2} dt dt' d\tau d\tau' \langle n_1(t)n_1(\tau)\rangle\langle n_2(t')n_2(\tau')\rangle Q(t-t')Q(\tau-\tau'). \tag{12}$$

It is convenient, in this context, to introduce the noise power spectrum in momentum space, $S_i(p)$, defined by

$$\langle n_i(t)n_i(\tau)\rangle=\frac{1}{2}\int dp S_i(p)e^{-2\pi iE(p)(t-\tau)}. \tag{13}$$

Assuming, as before, that T is much larger than the typical correlation intervals $t-t'$, $\tau-\tau'$, and using eq. (11) for $Q(p)$, then yields

$$\Delta S^2=\frac{T}{4}\int\frac{dp}{p}E(p)S_1(p)S_2(p)Q^2(p). \tag{14}$$

The optimal filtering is now determined by the choice (see [2] for details)

$$Q(p)=\lambda\frac{\gamma(p)\Omega(p)}{E^3(p)S_1(p)S_2(p)}, \tag{15}$$

where λ is an arbitrary normalization constant. With such a choice we finally arrive, from eq. (10) and (15), to the optimized signal-to-noise ratio:

$$SNR=\frac{\langle S\rangle}{\Delta S}=q_1q_2\frac{4H_0^2}{5\pi^2}\left[T\int\frac{dp}{pE^5(p)}\frac{\gamma^2(p)\Omega^2(p)}{S_1(p)S_2(p)}\right]^{1/2}. \tag{16}$$

It must be noted, at this point, that the functions $S_i(p)$ and $\gamma(p)$ appearing in the above equation are different, for a massive background, from the usual noise power spectrum $\tilde{S}_i(f)$, and overlap function $\tilde{\gamma}(f)$, conventionally used in the experimental analysis of gravitational antennas. Indeed, $\tilde{S},\tilde{\gamma}$ are defined as Fourier transforms of the frequency $f=E(p)$, so that (see for instance eq. (13)):

$$\int df\ \tilde{S}_i(f)e^{-2\pi ift}=\int dp\ S_i(p)e^{-2\pi iE(p)t},$$
$$\int df\ \tilde{\gamma}(f)e^{-2\pi ift}=\int dp\ \gamma(p)e^{-2\pi iE(p)t}, \tag{17}$$

from which

$$S_i(p) = (df/dp)\tilde{S}_i(f), \qquad \gamma(p) = (df/dp)\tilde{\gamma}(f). \tag{18}$$

By introducing into eq. (16) the known, experimentally meaningful variables $\tilde{S}_i, \tilde{\gamma}$, and using $f = E(p) = (m^2 + p^2)^{1/2}$, we thus arrive at the final expression:

$$SNR = q_1 q_2 \frac{4H_0^2}{5\pi^2}\left[T\int_0^{p_1} \frac{d\ln p}{(m^2+p^2)^{5/2}} \frac{\Omega^2(p)\ \tilde{\gamma}^2(\sqrt{m^2+p^2})}{\tilde{S}_1(\sqrt{m^2+p^2})\tilde{S}_2(\sqrt{m^2+p^2})}\right]^{1/2}. \tag{19}$$

This equation represents the main result of this paper. For any given massive spectrum $\Omega(p)$, and any pair of detectors with noise $\tilde{S}_i$ and overlap $\tilde{\gamma}$, the above equation determines the range of masses possibly compatible with a detectable signal ($SNR \gtrsim 1$), as a function of their coupling q_i to the detectors.

For $m = 0$ we have $p = f$, and we recover the standard relativistic result [2], modulo a different normalization of the overlap function. For $m \neq 0$ we shall assume, as discussed at the beginning of this paper, that the mass lies within the sensitivity and overlapping band of the two detectors, i.e. $\tilde{\gamma}(m) \neq 0$, and $\tilde{S}_i(m)$ is near the experimental minimum. Also, let us assume that the non-relativistic branch of the spectrum, $0 < p < m$, is near to saturate the critical density bound $\Omega < 1$, and thus dominates the total energy density of the background (the contribution of the relativistic branch $p > m$, if present, is assumed to be negligible).

To estimate the integral of eq. (19), in such case, we can thus integrate over the non-relativistic modes only. In that range, we will approximate $\tilde{S}_i$ and $\tilde{\gamma}$ with their constant values at $f = m$. Assuming that the spectrum $\Omega(p)$ avoids infrared divergences at $p \to 0$ (like, for instance, a blue-tilted spectrum $\Omega(p) \sim (p/p_1)^\delta$, with $\delta > 0$), we define

$$\int_0^m d\ln p\ \Omega^2(p) = \Omega_x^2, \tag{20}$$

where $\Omega_x \leq 1$ is a constant, possibly not very far from unity, and we finally arrive at the estimate

$$SNR \simeq q_1 q_2 \frac{4H_0^2\Omega_x}{5\pi^2}\left[\frac{T\ \tilde{\gamma}^2(m)}{m^5\tilde{S}_1(m)\tilde{S}_2(m)}\right]^{1/2}. \tag{21}$$

Following [2], the background can be detected, with a detection rate γ, and a false alarm rate α, if

$$SNR \geq \sqrt{2}\left(\mathrm{erfc}^{-1}2\alpha - \mathrm{erfc}^{-1}2\gamma\right). \tag{22}$$

For a first qualitative indication, let us consider the ideal case in which the two detectors are coincident and coaligned, i.e. $\tilde{\gamma} = 1$, $\tilde{S}_1 = \tilde{S}_2 = \tilde{S}$, $q_1 = q_2 = q$, and the massive stochastic background represents a dominant component of dark matter, i.e. $\Omega_x h_{100}^2 \sim 1$ (where $h_{100} = H_0/(100 \text{ km sec}^{-1} \text{ Mpc}^{-1})$ reflects the usual uncertainty in the present value of the Hubble parameter H_0). In such a case eq. (21), for an observation time $T = 10^8$ sec, a detection rate $\gamma = 95\%$, a false alarm rate $\alpha = 10\%$, gives the condition:

$$m^{5/2}\tilde{S}(m) \lesssim \frac{q^2}{3\pi^2} 10^{-31} \text{Hz}^{3/2}. \tag{23}$$

We will use here, for a particular explicit example, the analytical fit of the noise power spectrum of VIRGO, which in the range from 1 Hz to 10 kHz can be parametrized as [10]:

$$\tilde{S}(f) = 10^{-44}\text{sec}\Bigg[3.46 \times 10^{-6} \left(\frac{f}{500 \text{ Hz}}\right)^{-5} + 6.60 \times 10^{-2} \left(\frac{f}{500 \text{ Hz}}\right)^{-1} + 3.24 \times 10^{-2} + 3.24 \times 10^{-2} \left(\frac{f}{500 \text{ Hz}}\right)^{2}\Bigg]. \tag{24}$$

The intersection of this spectrum with the condition (23), in the plane $\left\{\log \tilde{S}, \log m\right\}$, is shown in Fig. 1 for three possible values of q^2. The allowed mass window compatible with detection is strongly dependent on q^2, and closes completely for $q^2 < 10^{-7}$, at least at the level of the noise spectrum used for this example. We should then consider two possibilities.

If the spectrum $\Omega(p)$ of eq. (19) refers to the spectrum of scalar metric fluctuations, induced on very small sub-horizon scales by an inhomogeneous, stochastic background of dark matter, then $q^2 = 1$ (since the detectors are geodesically coupled to metric fluctuations). In that case the detectable mass window extends over the full band from 1 Hz to 10 kHz, i.e from 10^{-15} to 10^{-11} eV.

If, on the contrary, scalar metric fluctuations are negligible on such small scales, and $\Omega(p)$ refers to the spectrum of the scalar background field itself, directly coupled to the detector through the scalar charge q, then this coupling is strongly suppressed in the mass range of Fig. 1, which corresponds to scalar interactions in the range of distance from 10^6 to 10^{10} cm. Otherwise, such scalar field would induce long range corrections to the standard gravitational forces that would be detected in the precise tests of Newtonian gravity and of the equivalence principle (see [11] for a complete compilation of the bounds on the coupling, as a function of the range).

Taking into account all possible bounds [11], it follows that, if the scalar coupling is universal (i.e. the induced scalar force is composition-independent), then the maximal allowed charge q^2 is around 10^{-7} from 1 to 10 Hz, and this upper bound grows proportionally to the mass (on a logarithmic scale) from 10 to 10^4 Hz. Composition-dependent couplings are instead more strongly constrained by Eotvos-like experiments: the maximal allowed value of q^2 scales like in the previous case, approximately, but the bounds are one order of magnitude stronger.

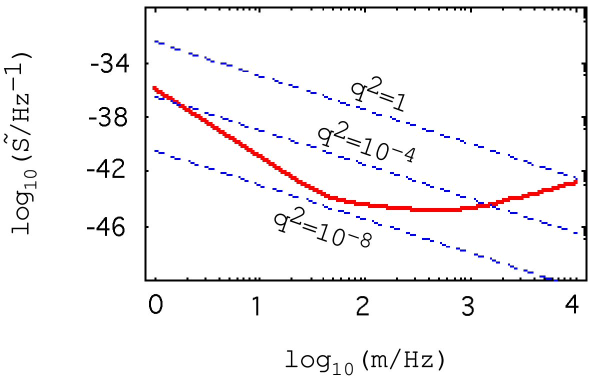

FIG. 1. *The bold curve corresponds to the noise power spectrum of VIRGO given in eq. (24). The thin, dashed lines represent the minimal sensitivity required for the detection of the background at the 90% confidence level (i.e., $SNR \simeq 2.5$), according to eq. (23). The mass window compatible with detection corresponds to the range of frequency for which $\tilde{S}$ is below a given dashed line.*

By inserting into the condition (23) the gravitational bounds on q^2 we are led to the situation illustrated in Fig. 2. A scalar background of nearly critical density, non-universally coupled to macroscopic matter, turns out to be only marginally compatible with detection (at least, in the example illustrated in this paper), since the line of maximal q^2 is just on the wedge of the noise spectrum (24). If the coupling is instead universal (for instance, like in the dilaton model discussed in [8]), but the scalar is not exactly massless, then there is a mass window open to detection, from 10^{-14} to 10^{-12} eV.

It seems appropriate to recall, at this point, that it is not impossible to produce a cosmic background of light, non-relativisic particles that saturates today the critical energy bound, as shown by explicit examples of spectra obtained in a string cosmology context [12]. Such particles, typical of string cosmology, are in general very weakly coupled to the total mass of the detector (like the dilatons, if they are long range, and the charge of the antenna is composition-dependent), or even completely decoupled (like the axions, since the total axionic charge is zero for a macroscopic, unpolarized antenna). Nevertheless, it is important to stress that they could generate a spectrum of scalar metric fluctuations, gravitationally coupled to the detector, which follows the same non-relativistic behaviour of the original spectrum. We know, for instance, that in cosmological models based on the low-energy string effective action, the variable representing the dilaton fluctuations exactly coincides with the scalar part of the metric fluctuations (at least in an appropriate gauge [13]), and that the associated spectra also coincide.

In view of the above discussion, the results illustrated in Fig. 1 and Fig. 2 suggest a new

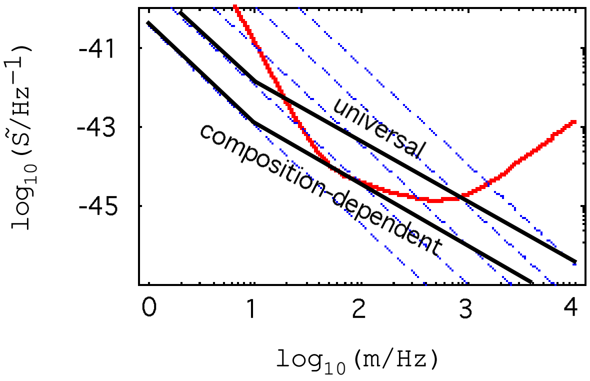

FIG. 2. *The noise spectrum of Fig. 1 is compared with the maximal values of* q^2 *(as a function of mass) allowed by gravitational tests, in two cases: composition-dependent and composition-independent scalar interactions. The thin dashed lines corresponds, from left to right, to* $q^2 = 10^{-8}$, 10^{-7}, 10^{-6}, 10^{-5}, 10^{-4}. *The region compatible with a detectable signal is above the noise spectrum and below the bounds given by the gravitational experiments.*

possible application of gravitational antennas, which seems to be interesting. Already at the present level of sensitivity, the gravitational detectors could be able to explore the possible presence of a light, massive component of dark matter, in a mass range that corresponds to their sensitivity band, in spite of the fact that such a massive background could be directly coupled to the total mass of the detector with a charge much weaker than gravitational, or only indirectly coupled, through the induced spectrum of scalar metric fluctuations.

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